지난 시간 → "efficient mechanism" (maximizes welfare of society)

이번 시간 → "optimal mechanism" (maximizes welfare of designer).

<u>Problem</u> For a single-obj., multi-bidder setting, design an auction game where the Nash equilibrium maximizes the expected utility of the seller. (= designer)

<u>Definitions</u> $N = \{1, 2, 3, \dots, n\}$: set of bidders. (w.l.o.g. seller = 0)

Each bidder $i \in N$ has a <u>value estimate</u> (VE) t_i drawn from p. distribution described by <u>PDF $f_i : [a_i, b_i] \to \mathbb{R}$</u>.

<u>CDF $F_i(t_i) = \int_{a_i}^{t_i} f_i(s_i)\, ds_i$</u>

t : vector of value estimates $[t_1, t_2, \dots, t_n]$.
= "type profile"

T: set of all possible value estimate vectors.

저번/이번과 notation 다른데!

Mechanism is a pair of outcome functions (p, x).

$$\begin{cases} p : T \to \mathbb{R}^n & \text{s.t. } p_i(t) = \text{Prob}(i \text{ gets obj.} \mid t) \\ x : T \to \mathbb{R}^n & \text{s.t. } x_i(t) = E[i \text{ pays } X \text{ to seller}] \end{cases}$$

cf) Recall Direct Revelation Mechanism (DRM).

$U_i(p, x, t_i)$: i's expected utility (i knows t_i!)

$U_0(p, x)$: seller's expected utility

<u>Assumptions</u> ① value estimate t_i drawn from f_i under <u>i.i.d.</u>

$\Rightarrow$ joint density function $f(t) = \prod_{j \in N} f_j(t_j)$.

② bidder i assesses probability distribution of t_j the same way as the seller. ($\Leftrightarrow f_j$ known)

$\Rightarrow$ joint density function $f_{-i}(t_{-i}) = \prod_{\substack{j \in N \\ j \neq i}} f_j(t_j)$.

③ bidders & sellers are <u>risk-neutral</u>

④ additively separable utility functions (= quasilinear environment)

cf) Recall: No "dictator" in quasilinear setting w/ WBB.

$$\Rightarrow U_i(p, x, t_i) = \int_{T_{-i}} \underbrace{\left(t_i p_i(t) - x_i(t)\right)}_{\text{얻는 가치의 exp. - 내야할 돈}} f_{-i}(t_{-i})\, dt_{-i}.$$

net utility for t.　　그 t일 PDF 값　　sum forall t.

$$U_0(p, x) = \int_T \left(t_0 \left(1 - \sum_{j \in N} p_j(t)\right) + \sum_{j \in N} x_j(t) \right) f(t)\, dt.$$

<u>Constraints</u> ① $\sum_{j\in N} p_j(t) \le 1$ and $p_i(t) \ge 0$ $\forall t\in T,\ i\in N.$

: $p_i(t)$ is a probability distribution over $N \cup \{0\}$.

② $U_i(p,x,t_i) \ge 0$ $\forall i\in N,\ \forall t_i \in [a_i, b_i].$

: "individual rationality" 개개인이 경매에 불참하지 않도록, 효용이 음이 아님.

③ $U_i(p,x,t_i) \ge \int_{T_{-i}} \left(t_i p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i) \right) f_{-i}(t_{-i})\, dt_{-i}$ 거짓말 $\forall i\in N,\ \forall t_i\in[a_i,b_i]$

$\forall s_i \in [a_i, b_i].$

: "revelation mechanism" 거짓말 방지책.

거짓말 시 효용이 정직할 때 효용보다 크지 않게. (= incentive compatibility)

$M = (p,x)$ is <u>feasible</u> if C-①, C-②, C-③ are satisfied.

<u>Problem</u> (restatement) $\underset{(p,x):\ \text{feasible DRM}}{\arg\max}\ U_0(p,x)$

<u>Def</u> $Q_i(p,t_i) \triangleq \text{Prob}(i \text{ gets obj.} \mid i\text{'s value estimate} = t_i)$

$= \int_{T_{-i}} \underbrace{p_i(t)}_{\text{Prob}(i \text{ gets obj} \mid t)} \underbrace{f_{-i}(t_{-i})}_{\text{Prob}(\text{value est.} = t_{-i})} dt_{-i}$: i 입장에서 물건 얻을 확률.

Lemma1 (Myerson Lem. 2).

(p,x) is feasible iff:

① if $s_i \le t_i$, then $Q_i(p,s_i) \le Q_i(p,t_i)$ $\forall i\in N,\ \forall s_i, t_i \in [a_i,b_i].$

$\Leftrightarrow$ 모든 진짜 value estimate ($= t_i$) 보다 낮은 모든 뻥카 ($= s_i$)에 대해,

i 입장에서 물건 얻을 확률이 뻥카가 더 크지 않다.

$\Leftrightarrow$ $Q_i(p,y)$는 $y\in[a_i,b_i]$에 대해 monotonically increasing.

② $U_i(p,x,t_i) = U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)\, ds_i.$ $\forall i\in N,\ \forall t_i\in[a_i,b_i]$

$\Leftrightarrow$ $U_i(p,x,t_i)$는 $t_i\in[a_i,b_i]$에 대해 monotonically increasing,

평균변화율은 i 입장에서 물건 얻을 확률을 구간에 대해 정적분한 값.

③ $U_i(p,x,a_i) \ge 0$ $\forall i\in N$

+② $\Rightarrow$ 경매에 참여 시 효용은 양보다 작지 않다. (individual rationality)

④ $\sum_{j\in N} p_j(t) \le 1$, $p_i(t) \ge 0$ $\forall i\in N,\ t\in T.$

$\Leftrightarrow$ constraint-①.

[feasible $\to$ Lem. 1 ①~④]

c-④ (revelation mechanism)

$$U_i(p,x,t_i) \geq \int_{T_{-i}} (\underbrace{t_i}_{s_i + (t_i - s_i)} p_i(t_{-i},s_i) - x_i(t_{-i},s_i)) f_{-i}(t_{-i})\, dt_{-i}.$$

$$= \int_{T_{-i}} (s_i p_i(t_{-i},s_i) - x_i(t_{-i},s_i)) f_{-i}(t_{-i})\, dt_{-i}$$

$$+ (t_i - s_i) \int_{T_{-i}} p_i(t_{-i},s_i) f_{-i}(t_{-i})\, dt_{-i}$$

$$= U_i(p,x,s_i) + (t_i - s_i) Q_i(p,s_i)$$

$\Rightarrow$ i) $U_i(p,x,t_i) - U_i(p,x,s_i) \geq (t_i - s_i) Q_i(p,s_i)$

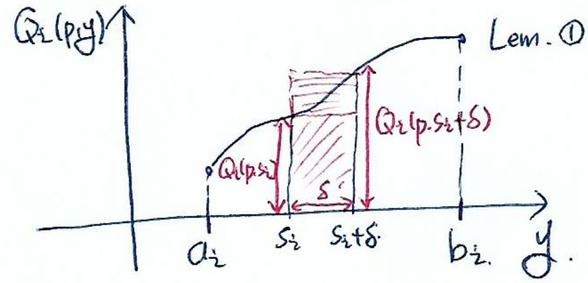
ii) switch s_i, t_i. $U_i(p,x,t_i) - U_i(p,x,s_i) \leq (t_i - s_i) Q_i(p,t_i)$

$\Rightarrow$ $(t_i - s_i) Q_i(p,s_i) \leq (t_i - s_i) Q_i(p,t_i)$

$\therefore$ Lem 1-① $Q_i(p,s_i) \leq Q_i(p,t_i)$ $\forall s_i \leq t_i$.

Now, choose $t_i = s_i + \delta$ for $\delta > 0$:

$$Q_i(p,s_i)\cdot\delta \leq U_i(p,x,s_i+\delta) - U_i(p,x,s_i) \leq Q_i(p,s_i+\delta)\cdot\delta. \quad \forall \delta > 0.$$



"Riemann integrable"

$\therefore$ Lem 1-② $U_i(p,x,t_i) = U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)\, ds_i$.

Moreover, c-② (individual rationality)

$U_i(p,x,t_i) \geq 0$ $\forall t_i \in [a_i, b_i]$ $\rightsquigarrow$ choose $t_i = a_i$.

$\therefore$ Lem 1-③ $U_i(p,x,a_i) \geq 0$.

[Lem. 1 ①~④ → feasible]

Recall: $Q_i(p, s_i) \geq 0$ $\because$ is a probability. $\forall s_i \in [a_i, b_i]$

$$\Rightarrow \int_{a_i}^{t_i} Q_i(p, s_i)\, ds_i \geq 0 \quad \forall t_i \in [a_i, b_i].$$

$$\therefore \underset{\text{Lem. 1-②}}{U_i(p, x, t_i) =} \underbrace{U_i(p, x, a_i)}_{\geq 0 \,(\text{Lem. 1 ③})} + \underbrace{\int_{a_i}^{t_i} Q_i(p, s_i)\, ds_i}_{\geq 0} \geq 0 \quad \forall t_i \in [a_i, b_i] \quad (c\text{-②})$$

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proving c-③ $\{U_i(p, x, t_i) \geq U_i(p, x, s_i) + (t_i - s_i) Q_i(p, s_i)\}$:

case 1: $s_i \leq t_i$.

$$U_i(p, x, t_i) = U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, r_i)\, dr_i \quad (\because \text{Lem. 1-②})$$

$$\geq U_i(p, x, s_i) + \int_{s_i}^{t_i} Q_i(p, s_i)\, dr_i \quad (Q_i(p, r_i) \geq Q_i(p, s_i) \text{ Lem. 1-①})$$

$$= U_i(p, x, s_i) + (t_i - s_i) Q_i(p, s_i)$$

case 2: $s_i > t_i$.

$$U_i(p, x, t_i) = U_i(p, x, s_i) - \int_{t_i}^{s_i} Q_i(p, r_i)\, dr_i.$$

$$\cdots \geq U_i(p, x, s_i) + (t_i - s_i) Q_i(p, s_i).$$

$\square$

Lemma 2 (Myerson Lem 3).

Let $p^*: T \to \mathbb{R}^n$, $x^*: T \to \mathbb{R}^n$ s.t.

$$\begin{cases} p^* = \underset{p \text{ satisfies Lem. 1-①, ④.}}{\arg\max} \displaystyle\int_T \left(\sum_{i \in N} \left(t_i - t_0 - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) p_i(t) \right) f(t)\, dt. \\ x_i^*(t) = \underbrace{t_i p_i(t)}_{\text{얻은가치의 기댓값}} - \displaystyle\int_{a_i}^{t_i} p_i(t_{-i}, s_i)\, ds_i \end{cases}$$

Then, (p^*, x^*) is optimal.

proof

$$U_0(p,x) = \int_T \Big(t_0\big(1-\sum_{j\in N} p_j(t)\big) + \sum_{j\in N} x_j(t)\Big) f(t)\,dt.$$

$t_0 = t_i + (t_0 - t_i)$

$$= \underbrace{\int_T t_0 f(t)\,dt}_{①} + \underbrace{\sum_{i\in N}\int_T p_i(t)(t_i - t_0) f(t)\,dt}_{②}$$

$$+ \sum_{i\in N}\int_T \big(x_i(t) - p_i(t)t_i\big) f(t)\,dt.$$

$$\shortparallel$$

$$-\int_{a_i}^{b_i} U_i(p,x,t_i) f_i(t_i)\,dt_i$$

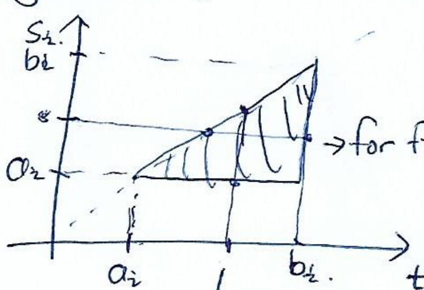
$$\overset{\text{Lem.1-④}}{=} -\int_{a_i}^{b_i}\Big(U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)\,ds_i\Big) f_i(t_i)\,dt_i.$$

$$= -U_i(p,x,a_i)\underbrace{\int_{a_i}^{b_i} f_i(t_i)\,dt_i}_{\text{확률총합} = 1.} - \underbrace{\int_{a_i}^{b_i}\int_{a_i}^{t_i} Q_i(p,s_i)\,ds_i\, f_i(t_i)\,dt_i.}_{= \int_{a_i}^{b_i}\int_{s_i}^{b_i} f_i(t_i) Q_i(p,s_i)\,dt_i\,ds_i}$$

(Fubini's theorem)

$f_i(t_i)Q_i(p,s_i) = h(t_i,s_i)$

integration region:



→ for fixed s_i, t_i range: $s_i \sim b_i$.

↳ for fixed t_i, s_i range: $a_i \sim t_i$.

$$= -U_i(p,x,a_i) - \int_{a_i}^{b_i}(1 - F_i(s_i))Q(p,s_i)\,ds_i$$

$$= -\underbrace{\;''\;}_{③} - \underbrace{\int_T (1 - F_i(t_i))\, p_i(t) f_{-i}(t_{-i})\,dt}_{④}$$

$$\Rightarrow U_0(p,x) = \underbrace{\int_T \Big(\sum_{i\in N}\Big(t_i - t_0 - \frac{1-F_i(t_i)}{f_i(t_i)}\Big) p_i(t)\Big) f(t)\,dt.}_{\text{independent of } x.}\ {}_{②+④}$$

$$+ \underbrace{\int_T t_0 f(t)\,dt}_{\text{constant.}}\ {}_{①} - \sum_{i\in N} U_i(p,x,a_i)\ {}_{③}$$

$\Downarrow$

(Let $x_i^*(t) = t_i p_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)\,ds_i.$)

$$\cdots = \sum_{i\in N}\int_{T_{-i}}\big(t_i p_i(t) - x_i(t)\big) f_{-i}(t_{-i})\,dt_{-i}.\Big|_{t_i = a_i}$$

> cancel.

$$= \underbrace{\sum_{i\in N}\int_{T_{-i}} \underbrace{\int_{a_i}^{t_i} p_i(t_{-i}, s_i)\,ds_i}_{=0} \cdot f_{-i}(t_{-i})\,dt_{-i}.\Big|_{t_i = a_i}}_{=0}$$

$= 0$. minimize.

$(\because U_i \geq 0)$

Note: such an x^* satisfies Lem. 1-②, ③. (omitted)

Now, p should maximize $\boxed{}$

$\square$

Corollary. (Revenue-Equivalence Thm)

The seller's utility $U_0(p, x)$ from a feasible mechanism (p, x) is completely determined by p and the numbers $U_i(p, x, a_i)$ $\forall i \in N$.

Assumption - ⑤. Regularity. Function $c_i : [a_i, b_i] \to \mathbb{R}$ defined as

$$c_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

is a monotone strictly increasing function of t_i, $\forall i \in N$.

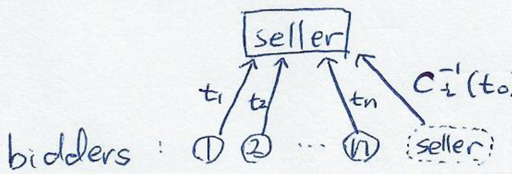
cf) $c_i(t_i)$: "virtual surplus" of buyer
= "marginal revenue" 한계 수익.

$$\left(\text{willingness to pay}\right) - \frac{\text{demand}}{\text{cost decr.}} = \Delta\ \text{revenue}.$$

... turns out,

$$p^*(t) = \begin{cases} \vec{0} & \text{if } t_0 > \max_{i \in N} c_i(t_i) \quad \text{(seller keeps item)} \\ [0, \ldots, 0, \underset{\substack{\uparrow \\ \text{tiebreaking} \\ \text{winner} \\ \text{w/ max } c_i(t_i)}}{1}, 0, \ldots, 0] & \text{otherwise} \end{cases}$$

$$x^*_i(t) = \begin{cases} \inf\{s_i \mid c_i(s_i) \geq 0,\ c_i(s_i) \geq c_j(s_j)\ \forall j \neq i\} & \text{if } p_i(t) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

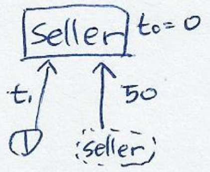


$c_i^{-1}(t_0)$: if symmetric bidders ($[a_i, b_i] = [a_j, b_j]$, $f_i = f_j$).

Now, consider : $p^*(t) = \vec{0}$.

$U_i(p, x, t) = U_0(p, x) = 0$ ⇝ NOT efficient.

e.g. $n = 1$



$c_i^{-1}(t_0) = 50$

$\Uparrow$

$$\left.\begin{array}{l} t_i \in [0, 100] \\ f_i(t_i) = 1/100 \end{array}\right\} \Rightarrow c_i(t_i) = t_i - \frac{1 - \frac{t_i}{100}}{1/100} = 2t_i - 100$$

if $0 < t_1 < 50$, Σ utility $= 0$.
However, max Σ Utility $= t_1$.
NOT efficient.

Concluding remarks

Myerson considers a more general setting.

- $v_i(t) = t_i + \sum_{\substack{j \in N \\ j \neq i}} e_j(t_j)$: value estimate revision function e_i로 물건의 객관적 가치 반영.

- Non regular $c_i(t_i)$:

$$\begin{cases} \bar{p}_i(t) = \begin{cases} 1/|M(t)| & \text{if } i \in M(t) \\ 0 & \text{otherwise.} \end{cases} \\ \bar{x}_i(t) = (\text{Lem. 2 - } x \text{ condition}) \end{cases}$$

$M(t)$ 은 $\{ i \mid t_0 \le \bar{c}_i(t_i) = \max_{j \in N} \bar{c}_j(t_j) \}$

def of $\bar{c}_i(t_i)$ technical.

is optimal.

- Non-i.i.d. value estimates: linear programming

- Implementation: compute $\bar{c}_j$ (or c_j) and x_i from f_i (and e_i)

  $\Rightarrow$ easy to compute.