Auctions vs Negotiations

English Auction

- English auction : the price rises continuously until only one bidder remains. At every price all bidders know how many other bidders remain active.
- Auction with a reserve price : the seller makes a final take-it-or-leave-it offer equal to the reserve price to the final bidder, if the final bid is below the reserve price.
- Absolute auction : an auction with no reserve price.

Simple Problem

- Seller A
  value is 0, has one “serious” potential buyer ($\geq$ seller’s value)
  whose value is drawn from uniform dist. on [0, 1].
  negotiate to offer take-it-or-leave-it price
- Seller B
  value is 0, has two “serious” bidders,
  whose values are drawn from same dist. independently.
  only run an English auction with on reserve.
- Both Parties in both case are risk neutral.

Simple Problem

- Consider a seller with constant marginal cost of 0.
- linear demand curve $p = 1 - q$
- Define Revenue as price times quantity
- $MR = 1 - 2q$ which is 0 at quantity of .5

Expected Revenue (negotiate with one)

- Assume that:
  bidder j receives a private, independent signal t_j,
  distributed with pdf $f(t_j)$, cdf $F(t_j)$,
  private value $v(t_j)$, quantity $q(t_j) \equiv 1 - F(t_j)$.

- $MR(t_j) \equiv \frac{d}{dq(t_j)} v(t_j) q(t_j) = \frac{-1}{f(t_j)} \frac{d}{dt_j} v(t_j)[1 - F(t_j)]$

- Expected revenue : the area under the MR curve for all the values in excess of the take-it-or-leave-it price

Expected Revenue (negotiate with one)

- At optimal take-it-or-leave-it price (where MR = 0), in expectation, the seller may be thought of as the MR of the buyer when it is positive, 0 otherwise.
- Expected revenue is $E\{\max(MR(t_1), 0)\}$.

Expected Revenue (auction with two)

- Assume that the value of the ”underbidder” is $v(t_i)$,
  the value of the eventual winner is $v(t_j)$.
- In the auction, learn $v(t_i)$, which is lower bound of $v(t_j)$.
- The seller will earn $v(t_i)$, which is expected MR of the winner.
- Averaging over all possible $v(t_i)$,
  expected revenue equals the expected MR of the winner.

Expected Revenue (auction with two)

- With convectional assumption that MR curve is downward sloping,
  which implies that buyer with the higher value who will actually win the auction has the higher MR.
- Expected revenue is $E\{\max(MR(t_1), MR(t_2))\}$.

Expected Revenue (auction with two)

- $E\{\max(MR(t_1), MR(t_2))\}$ vs $E\{\max(MR(t_i), 0)\}$.
- If $MR(t_1) \geq 0$, the first expression is larger.
- If $MR(t_1) < 0$, use “serious bidder” assumption.
  Since the lowest possible $v(t_2)$ is 0, expectation of $MR(t_2)$ is 0.
- The first expression is the expectation of the maximum of two terms, one of which has an expected value of 0.
  Therefore, the first expression is larger.

Model With Independent Private Values

- Extend to the analysis to compare “a seller with N symmetric bidders in an auction with reserve price” to “one with N+1 bidders and no reserve price.
- By same analysis, expected revenue are $E\{\max(MR(t_1), \cdots, MR(t_N), 0)\}$ and $E\{\max\left(MR(t_1), \cdots, MR(t_{N+1})\right)\}$
- Since expectation of $MR(t_{N+1}) = 0$, the auction with the extra bidder yields a higher expected revenue.

The General Model

- Let t_j be bidder j's private signal. (no need to be independent)
- WLOG, normalize so that $0 \leq t_j \leq 1\ \forall j$ and seller's value is 0.
- $\boldsymbol{T}$ represent the vector $(t_1, \cdots, t_{N+1})$,
  $\boldsymbol{T}_{-j}$ all of the elements of $\boldsymbol{T}$ other than t_j
  $\overline{\boldsymbol{T}} \equiv \boldsymbol{T}_{-(N+1)}$ and $\overline{\boldsymbol{T}}_{-j}$ similarly.

The General Model

- $f(t_j|\boldsymbol{T}_{-j})$ for the conditional density of t_j given $\boldsymbol{T}_{-j}$,
  (assume positive and finite for all t_j and $\boldsymbol{T}_{-j}$)
  $F(t_j|\boldsymbol{T}_{-j})$ for the cdf of $f(t_j|\boldsymbol{T}_{-j})$.

- Let $v_j(\boldsymbol{T})$ be the value of the asset to bidder j,
  $\overline{v_j}(\overline{\boldsymbol{T}}) \equiv E_{t_{N+1}}\{v_j(\boldsymbol{T})\} = \int_0^1 v_j(\boldsymbol{T}) f(t_{N+1}|\overline{T}) dt_{N+1}$ be the expectation of $v_j(\boldsymbol{T})$ conditional on $\overline{\boldsymbol{T}}$.
  (Note that $E\{u(X,Y)|x\} = \int_{-\infty}^{\infty} u(x,y) f(y|x) dy = E_y\{u(X,Y)\}$.)

- Higher signals imply higher expected values and signals are affiliated,
  $\frac{\partial v_j(\boldsymbol{T})}{\partial t_j} > 0$, $\frac{\partial v_i(\boldsymbol{T})}{\partial t_j} \geq 0$, and $t_j \geq t_i \Longrightarrow v_j(\boldsymbol{T}) \geq v_i(\boldsymbol{T})$ for all $i, j, \boldsymbol{T}$

- While signals are private information, functions $v_j(\boldsymbol{T})$ and $f(t_j|\boldsymbol{T}_{-j})$ are common.

The General Model

- $MR_j(\boldsymbol{T}) \equiv \frac{-1}{f\left(t_j \middle| \boldsymbol{T}_{-j}\right)} \frac{d}{dt_j}\left[v_j(\boldsymbol{T})\left[1 - F\left(t_j \middle| \boldsymbol{T}_{-j}\right)\right]\right]$ and

  $\overline{MR}_j(\overline{\boldsymbol{T}}) \equiv \frac{-1}{f\left(t_j \middle| \overline{\boldsymbol{T}}_{-j}\right)} \frac{d}{dt_j}\left[v_j(\overline{\boldsymbol{T}})\left[1 - F\left(t_j \middle| \overline{\boldsymbol{T}}_{-j}\right)\right]\right]$.

- Assumptions

  A.1 : Downward-Sloping MR

  A.2 : Serious Bidders

  A.3 : Symmetry, $v_i\left(t_1, \cdots, t_i, \cdots, t_j, \cdots\right) = v_j\left(t_1, \cdots, t_j, \cdots, t_i, \cdots\right) \forall i, j, \boldsymbol{T}$.

The General Model

- Lemma 1 : The expected revenue from an absolute English auction with N+1 bidders equals $E_{\boldsymbol{T}}\left\{\max\left(MR_1(\boldsymbol{T}), \cdots, MR_{N+1}(\boldsymbol{T})\right)\right\}$.
- Write $(x, \boldsymbol{T}_{-j})$ for the vector $\boldsymbol{T}$ but with the jth element replaced by x.
- Proof.
  If bidder j has the highest and bidder i has the second-highest signal,
  then bidder j will win the auction at the price $v_i\left(t_i, \boldsymbol{T}_{-j}\right)$
  which is equals to $v_j\left(t_i, \boldsymbol{T}_{-j}\right)$ by symmetry.

$$v_j\left(t_i, \boldsymbol{T}_{-j}\right) = E_{t_j}\{MR_j(\boldsymbol{T}) | t_j \geq t_i, \boldsymbol{T}_{-j}\}$$

The General Model

- Lemma 2 : The expected revenue from an English auction with N risk-neutral bidders followed, after the N-1 low bidders have quit, by an optimally chosen take-it-or-leave-it offer to the remaining bidder, equals $E_{\overline{\boldsymbol{T}}}\{\max(\overline{MR}_1(\overline{\boldsymbol{T}}), \cdots, \overline{MR}_N(\overline{\boldsymbol{T}}), 0)\}$.
- Write $(x, \overline{\boldsymbol{T}}_{-j})$ for the vector $\overline{\boldsymbol{T}}$ but with the jth element replaced by x.
- Proof.
  i leaves at price $\bar{v}_i(t_i, \overline{\boldsymbol{T}}_{-j})$ equals $\bar{v}_j(t_i, \overline{\boldsymbol{T}}_{-j})$. The seller choose a take-it-or-leave-it offer for the last bidder of $\bar{v}_j(\hat{t}, \overline{\boldsymbol{T}}_{-j})$ where $\hat{t} \geq t_i$.
  If $t_j \geq \hat{t}$, then the seller will receive $\bar{v}_j(\hat{t}, \overline{\boldsymbol{T}}_{-j}) = E_{t_j}\{\overline{MR}_j(\overline{\boldsymbol{T}}) | t_j \geq \hat{t}, \bar{T}_{-j}\}$.
  If $t_j < \hat{t}$, then the seller will receive 0.

The General Model

- Thoerem 1 : Expected revenue from an absolute English auction with N+1 bidders exceed expected revenue from an English auctions with N bidders followed by a take-it-or-leave-it offer to the last remaining bidder if either (i) bidders’ values are private; or (ii) bidders’ signals are affiliated.

The General Model

- Consider the cases in which bidder j, the highest of the first N bidders, has a positive or negative $\overline{MR_j}$. (His value exceeds the optimal reserve price or not.)
- If $\overline{MR_j} < 0$, the expectation over t_{N+1} of MR_{N+1} equals bidder $N + 1$'s lowest possible value, which equals or exceeds 0 by A.2.
- If $\overline{MR_j} \geq 0$, affiliation implies that expectation of MR_j is greater than or equal to the expectation of $\overline{MR_j}$.

$$\int_0^1 v_j(\hat{t}, \boldsymbol{T}_{-j}) f(t_{N+1} | t_j \geq \hat{t}, \overline{\boldsymbol{T}}_{-j}) dt_{N+1}$$

$$\geq \int_0^1 v_j(\hat{t}, \boldsymbol{T}_{-j}) f(t_{N+1} | t_j = \hat{t}, \overline{\boldsymbol{T}}_{-j}) dt_{N+1}$$