Algorithmic Pricing via Virtual Valuations

Shuchi C., Jason D. H., Robert D. K.

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Algorithmic Pricing

Given:

- List of Prices for all possible allocations to a consumer
- Consumer's preference indicates a most desired allocation

Goal:

- Take an instance given by a class of allowable pricings and a set of consumers,
  and compute the pricing **maximizing**(or approximately maximizing) a specific objective

Algorithmic Pricing

- Problem 1: Bayesian Single-item Auction Problem (BSAP)
  - ➢Single item for sale,
  - ➢N consumers,
  - ➢Distribution F from which consumer valuations are drawn

  Goal: design seller optimal auction for F

Algorithmic Pricing

- Problem 2: Bayesian Unit-demand Pricing Problem (BUPP)
  - ➢Single unit-demand consumer,
  - ➢N items for sale,
  - ➢Distribution F from which the consumer's valuations for each item are drawn

  Goal: compute seller optimal item-pricing for F

Notations

- Valuation vector $\mathbb{v} = (v_1, \ldots, v_n)$
  - ➢BSAP v_i: valuation of consumer i for the single item
  - ➢BUPP v_i: valuation of single consumer for item i

  v_i drawn independently from distribution F_i over range $[l_i, h_i]$

- $\mathbb{v}_{-i}$: all valuations except the ith
- $\mathbb{F} = F_1 \times \cdots \times F_n$
- $f_i(v_i)$: probability density of v_i

Definitions

- Monotone Hazard Rate

Given a distribution F with density f, the *hazard rate* of F:

$$\frac{f(v)}{1 - F(v)}$$

is monotonically non-decreasing function of v.

Definitions

- Regularity

Given a distribution F with density f is regular if

$$v - \frac{1 - F(v)}{f(v)}$$

is monotonically non-decreasing for all v.

When each F_i is regular, then $\mathbb{F}$ is regular.

BSAP

- single item for sale
- n-consumers with values given by $\mathbb{V}$
- v_i from a distribution F_i
- revenue $\mathcal{R}^{\mathcal{A}}$

BSAP

- Virtual valuations

Virtual valuation of bidder i with valuation v_i drawn from F_i is

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

The virtual surplus of a BSAP is the virtual valuation of the winner.

BSAP

- Myerson's Theorem

Any incentive-compatible auction $\mathcal{A}$ has expected revenue equal to its expected virtual surplus

➢ Maximizing revenue = Maximizing virtual surplus

➢ Set the price as take-it-or-leave-it price

$$\mathrm{p_i} = \phi_i^{-1}(v_i) \; where \; v_i = \max_{j \neq i} \max\left(\phi_j\left(v_j\right), 0\right).$$

➢ $\mathcal{R}^{\mathcal{M}}$: the revenue of Myerson's auction.

BSAP

- $\mathcal{R}^{\mathcal{A}}$: revenue of BSAP
- $\mathcal{v}$: reservation value of item
- When $\mathbb{F}$ is regular, $\mathcal{R}^{\mathcal{M}_v} + v \cdot \mathcal{X}(\mathcal{M}_v) \geq \mathcal{R}^{\mathcal{A}} + v \cdot \mathcal{X}(\mathcal{A})$ for all incentive-compatible auctions $\mathcal{A}$.

Fact: Virtual valuations satisfy $\phi_i(v_i) \leq v_i$

$\mathcal{X}(\mathcal{A})$ is the probability of item not sold

BUPP

- n-item for sale
- single consumer with unit-demand
- quasi-linear preferences given by vector $\mathbb{v}$.

$$u(v_1, \dots, v_n) = v_1 + \theta(v_2, \dots, v_n)$$

- v_i from a distribution F_i
- $\mathbb{p}$: price vector
- revenue $\mathcal{R}^{\mathbb{p}}$

$$\mathcal{R}^{\mathbb{p}} = \sum_i p_i \cdot \Pr_{v \sim F}\left[(v_i - p_i) = \max_{j<n}(v_j - p_j)\right]$$

BSAP and BUPP

- When $n = 1$ BSAP and BUPP are equal and the revenue is

$$p_1 = \phi_i^{-1}(0)$$

- For $n > 1$, the optimal auction $\mathcal{R}^{\mathcal{M}}$ for BSAP will obtain at least the revenue $\mathcal{R}^{\mathbb{p}}$ for any pricing $\mathbb{p}$ for BUPP

$$\mathcal{R}^{\mathcal{M}} \geq \mathcal{R}^{\mathbb{p}}$$

BSAP and BUPP

For a pricing $\mathbb{p}$ consider mechanism $\mathcal{A}_{\mathbb{p}}$:

- Allocate item to bidder i that maximizes $v_i - p_i$, with standard threshold payment
- Because of monotone allocation $\mathcal{A}_{\mathbb{p}}$ is truthful

$$\mathcal{R}^{\mathcal{A}_{\mathbb{p}}} \leq \mathcal{R}^{\mathcal{M}}$$

BSAP and BUPP

Given a valuation vector $\mathbb{v}$ and suppose that $\mathcal{A}_{\mathbb{p}}$ allocates the item to bidder i

- Minimum bid of the item is

$$p_i + \max(v_j - p_j, 0)$$

- revenue of $\mathcal{A}_{\mathbb{p}}$ with $\mathbb{v}$ is at least p_i
- revenue of pricing $\mathbb{p}$ is exactly p_i

$$\mathcal{R}^{\mathcal{A}_{\mathbb{p}}} \geq \mathcal{R}^{\mathbb{p}}$$

$$\boldsymbol{\mathcal{R}}^{\boldsymbol{\mathcal{M}}} \geq \mathcal{R}^{\mathcal{A}_{\mathbb{p}}} \geq \boldsymbol{\mathcal{R}}^{\mathbb{p}}$$

Computational model for BUPP

- Given $v = \max(0, v_{1/2})$, the pricing $\mathbb{p} = \mathbb{r}(v)$

$$\mathcal{R}^{\mathbb{p}} \geq \mathcal{R}^{\mathcal{M}}/3\,.$$

v_x: $\mathcal{X}\left(\mathbb{r}(v_x)\right) = x$

Computational model for BUPP

- Corollary 6

$$\mathcal{R}^{\mathcal{M}_v} + v \cdot \mathcal{X}(\mathcal{M}_v) \geq \mathcal{R}^{\mathcal{M}}$$

$\mathcal{R}^{\mathcal{M}_v} + v \cdot \mathcal{X}(\mathcal{M}_v) \geq \mathcal{R}^{\mathcal{A}} + v \cdot \mathcal{X}(\mathcal{A}) \Rightarrow$
$\mathcal{R}^{\mathcal{M}_v} + v \cdot \mathcal{X}(\mathcal{M}_v) \geq \mathcal{R}^{\mathcal{M}}$ (give $\mathcal{A} = \mathcal{M}$ and $\mathcal{X}(A) \geq 0$)

Computational model for BUPP

- Lemma 7

$$For\ \mathbb{p} = \mathbb{r}(v), \mathcal{R}^{\mathbb{p}} \geq \left(1 - \mathcal{X}(\mathbb{p})\right) \cdot v$$

Remark that $1 - \mathcal{X}(\mathbb{p})$ is the probability of item being sold

Computational model for BUPP

- Lemma 8

$$For\ any\ \mathbb{p}, \mathcal{R}^{\mathbb{p}} \geq \mathcal{X}(\mathbb{p}) \cdot \sum_i p_i q_i$$

$$q_i = 1 - F_i(p_i)$$

Computational model for BUPP

- Lemma 9

$$\text{Under regularity, for any } \mathbb{p} \geq \mathbb{r}(0)$$
$$\text{and any incentive} - \text{compatible auction } \mathcal{A}$$
$$we\ have\ \mathcal{R}_{\mathbb{p}}^{\mathcal{A}} \leq \sum_i p_i q_i$$

Computational model for BUPP

- Lemma 9

$$\text{Under regularity, for any } \mathbb{p} \geq \mathbb{r}(0)$$
$$\text{and any incentive} - \text{compatible auction } \mathcal{A}$$
$$we\ have\ \mathcal{R}_{\mathbb{p}}^{\mathcal{A}} \leq \sum_i p_i q_i$$

Let mechanism $\mathcal{A}'$ sell if and only if $v_i \geq p_i$.
Then $\mathcal{R}_{\mathbb{p}}^{\mathcal{A}'} = \mathcal{R}^{\mathcal{A}'}$ and $\mathcal{R}_{\mathbb{p}}^{\mathcal{A}} \leq \mathcal{R}_{\mathbb{p}}^{\mathcal{A}'}$.

Since $\mathcal{R}^{\mathcal{A}'}$ revenue is less than the optimal auction $\sum_i p_i q_i$, the lemma holds.

Computational model for BUPP

- Corollary 10

Under regularity,
any auction $\mathcal{A}$ *and any pricing* $\mathbb{p} \geq \mathbb{r}(0)$ *satisfies*

$$\mathcal{X}(\mathbb{p}) \cdot \mathcal{R}_{\mathbb{p}}^{\mathcal{A}} \leq \mathcal{R}^{\mathbb{p}}$$

- Corollary 11

Under regularity,
for any $v \geq 0, \mathbb{p} = \mathbb{r}(v)$ satisfies $\mathcal{X}(\mathbb{p}) \cdot \mathcal{R}^{\mathcal{M}_v} \leq 3\mathcal{R}^{\mathbb{p}}$

Computational model for BUPP

- Lemma 12

Under regularity, with $\mathcal{v}_{1/2} \geq 0$,
$for\ any\ \mathcal{v} \geq 0, \mathbb{p} = \mathbb{r}(\mathcal{v}_{1/2})$ satisfies $\mathcal{X}(\mathbb{p}) \cdot \mathcal{R}^{\mathcal{M}_{\mathcal{v}}} \leq \mathcal{R}^{\mathbb{p}}$

$$\mathcal{R}^{\mathcal{M}} \leq \mathcal{R}^{\mathcal{M}_{\mathcal{v}_x}} + \mathcal{v}_x x \leq \frac{1}{x}\mathcal{R}^{\mathbb{p}} + \mathcal{v}_x x \leq \frac{1}{x}\mathcal{R}^{\mathbb{p}} + \frac{x}{1-x}\mathcal{R}^{\mathbb{p}} = 3\mathcal{R}^{\mathbb{p}}$$

$$\Rightarrow \mathcal{R}^{\mathbb{p}} \geq \mathcal{R}^{\mathcal{M}}/3$$

QnA