Price of Anarchy for Auction Revenue

Author: Jason Hartline, Darrell Hoy, Sam Taggart

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After the draw, she will determine a subsequent action according to the strategy $s_i : V_i \to A_i$.

Bayesian Mechanisms

Definition: Mechanism

A mechanism M is a pair of two function $\tilde{x}, \tilde{p} \in (A_1 \times \ldots \times A_n) \to \mathbb{R}^n$ where:

- $\tilde{x}_i(a_1, \ldots, a_n)$ denotes the probability where i-th agent gets the item.
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Agent i's utility is $\tilde{u}_i(a) = v_i \tilde{x}_i(a) - \tilde{p}_i(a)$.

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Simply speaking, it's in the order of Values (V_i), Sampling (F_i), Strategy (S_i), Action (A_i), Utility ($u_i(a)$), as in the agent perspective.

We want best strategy that works anytime, but even a tiny prison with two rooms don't have such. This is a gargantuan setting.

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So we work with **Bayes-Nash equilibrium:** For each agents i, the **strategy** is optimal within the **Mechanism** given other agent's **Value, Distribution, Actions, Strategies** are fixed.

You can find the correspondence for all five items. Yes, everything except my strategy is fixed.

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Everything is probabilistic BTW, so the profit is expectation of utility according to the distribution $F_1, \ldots, F_n$.

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This is the original definition of mechanism in Lecture 2. Throughout the lecture, we will reintroduce some of the concepts in Lecture 2 in a generalized form.

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Question: Why are people interested in maximizing this term?

Myerson introduced the following lemma while introducing the revenue equivalence theorem:

Lemma

Let f_i be the pdf of F_i and F_i be the cdf of F_i. Define the **Virtual value** $\phi(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$*. It follows that*
$Rev(M) = E_v[\sum_i p_i(v)] = E_v[\sum_i \phi_i(v_i)x_i(v)]$

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The proof for the special case can be found in Lecture 2 notes, I think?

Price Of Anarchy

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I recommend you to visit the Wikipedia page *Price of anarchy in auctions*. It looks really algorithm-ish.

Price Of Anarchy

Definition: Price of anarchy for revenue

Let $BNE(M, F)$ be the set of Nash equilibrium strategies per the distribution F. The **Bayesian price of anarchy for revenue** is defined as $\max_{F \in R, s \in BNE(M,F)} Rev(OPT_F)/Rev(M)$, where R is the set of regular distributions and OPT_F is the Bayesian revenue-optimal mechanism for value distribution F.

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Note that, we only consider cases where the agents can *leave if they want*: All agents are given an action to finish the auction with $u_i(a) = 0$. So you don't have to worry about niche cases where the term is negative.

I think, the term can be unbounded, though.

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What is the *individual monopoly reserves*? It means the seller can impose a certain threshold where she would not sell if the price is less than the **reserve value** r_i.

r_i may be different per agent.

We denote this mechanism FPA_r.

Price Of Anarchy

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Theorem

For any Bayes-Nash equilibrium strategy,
$Welfare(FPA_r) + Rev(FPA_r) \geq \frac{e-1}{e} Welfare(OPT_r)$.

Price Of Anarchy

From the theorem we can obtain $O(1)$ PoA for revenue and welfare.

Theorem

For any Bayes-Nash equilibrium strategy,
$Rev(OPT_r)/Rev(FPA_r) \leq \frac{2e}{e-1}$*, and*
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The proof of these facts is in the paper.

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They can be modeled as a flow, where each path incurs flow of its own probability.

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On the original problem, there are known bounds of PoA:

Theorem

If a latency function is a linear function the PoA is at most $\frac{4}{3}$.

Theorem

If a latency function is a polynomial of degree at most d, *the PoA is at most* $d+1$.

Thank you for your attention!