Bayesian Combinatorial Auctions

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Combinatorial Auction

- m items, n bidders.
- Each bidder i has valuation function $v_i : [m] \to \mathbb{R}_{\geq 0}$.
  - i.e. Each bidder has value on the **set of items**.
- Find a partition of $[m] = S_1 \sqcup \cdots \sqcup S_n$ such that the total social welfare $v_1(S_1) + \cdots + v_n(S_n)$ is maximized.

Welfare-maximizing Partition

General assumptions:

- $v_i(\emptyset) = 0$
- $S \subseteq T \implies v_i(S) \leq v_i(T)$.

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Even though we have full information about all bidders' valuation function, we face some general hardness.

Even if $v_1 = \cdots = v_n = f$ and f is submodular, it reduces to celebrated submodular multi-way partition. **(APX-hard)**

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Applying VCG-based pricing on x^* is known to be truthful, but finding such allocation requires exponential time.

Hardness Results

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Hardness Results

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- VCG is truthful, but highly inefficient. (Nisan, 1999)
- Moreover, even if all valuations are **submodular**, no polynomial-time truthful mechanism can approximate the optimal SW within the factor $m^{1/2-\varepsilon}$. (Dobzinski, 2011)
- Some *untruthful* mechanism may give some good approximation of optimal SW. (Nisan, 1999)

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Simplified auction model (eBay auction)

We still have much more things to do with a much simpler mechanism.

- m items, each in independent second-price auction
- n bidders bid in each auction simultaneously.
  - A bid is an m-dimensional vector of non-negative numbers.
- Their valuations are still set-valued functions.

Where the bidder i wins the items $S_i \subseteq [m]$, the social welfare is $\sum_{i=1}^{n} v_i(S_i)$.

Bayesian game

- Let $v = (v_1, \cdots, v_n)$ be the complete valuation profile. Define v_{-i} and (v_i, v_{-i}) in straightforward way.
- $B_i(v_i)$ is the bid result once v_i is drawn as the value. it's an m-dimensional vector. B_i's are called **bidding function**.
- $X_i(B(v))$ is set of allocated items to i, when v, B is determined.

Bayesian game

- Utility of player i is

$$
\begin{aligned}
& v_i(X_i(B(v))) - \sum_{j \in X_i(B(v))} \mathrm{SecondPrice}(B(v), j) \\
& = v_i(X_i) - \sum_{j=1}^{m} \max_{k \neq i} \left[B_k(v_k) \right]_j .
\end{aligned}
$$

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- However, here we don't know the ground-truth of the "value of a single item".
- Moreover, we'll leak some information – the *(finite independent)* probability distribution D_i of v_i as a public knowledge.
- A variety of pure-or-mixed Nash equilibriums may arise. Note that the only knob is on the bidding function B_i.

Bayes-Nash Equilibrium

Bayes-Nash Equilibrium (BNE)

$B = (B_1, \cdots, B_n)$ is called Bayes-Nash Equilibirum (w.r.t. D) when the bid $B_i(v_i)$ maximizes the utility of i, compared to any other bidding function $\tilde{B}_i(v_i)$
assuming other bidding strategy B_j is fixed, and v_j are drawn from the distribution D_j.

Price of Anarchy

We evaluate the price of anarchy for a BNE B:

- Expected Optimal Social Welfare

$$\mathrm{EO}(D) := \sum_{v:\text{possible all valuations}} D(v) \cdot \mathrm{OptimalSocialWelfare}(v)$$

- Expected social welfare

$$\mathrm{ESW}_D(B) := \sum_{v} D(v) \cdot \mathrm{SW}(B(v))$$

Where $D(v) := \prod_{i=1}^{n} D_i(v_i)$, and $\mathrm{SW}(\mathrm{B}(\mathrm{v}))$ is the social welfare obtained by running auctions with the bids $B(v)$.

Price of Anarchy

Given the distribution D, the **price of anarchy** is defined as

$$\mathrm{PoA}(D) := \max_{B\,:\,\textsf{BNE w.r.t } D} \left(\frac{\mathrm{EO}(D)}{\mathrm{ESW}(B)} \right).$$

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This demonstrates the worst-case approximation quality about **social welfare**. If we can restrict PoA within some constant bound, this justifies using eBay auction as an approximate for the sophisticated combinatorial auction.

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Necessary assumptions

Think of this example.

Table: Players and Values

item set	**bidder 1**	**bidder 2**
$\{1\}$	1	0

Social welfare is maximized when the item goes to the bidder 1 with truthful bids, obtaining the value 1.

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Social welfare is maximized when the item goes to the bidder 1 with truthful bids, obtaining the value 1.
However, if bidder 1 underbids to 0 and bidder 2 **overbids** to 1, we get stuck on the **pure Nash eqb** with social welfare 0. The PoA explodes to infinity!

Necessary assumptions

To mitigate this, we shall add the **no-overbidding** assumption. For any bidding vector $b_i = B_i(v_i)$ and $T \subseteq [m]$,

$$\sum_{j \in T} b_{ij} \leq v_i(T).$$

We still need more assumptions, especially on the *submodularity* of v_i.

Submodularity

Submodularity

A function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is called **submodular** if for any $S, T \subseteq [n]$,

$$f(S \cup T) \leq f(S) + f(T) - f(S \cap T).$$

equivalently, for any $j \notin S \subseteq T \subseteq [n]$,

$$f(T \cup \{j\}) - f(T) \leq f(S \cup \{j\}) - f(S).$$

In some sense, it generalizes the convexity in inclusion lattice.

Beyond the submodularity

Subadditivity

A function $f : [n] \to \mathbb{R}_{\geq 0}$ is called *subadditive* if for any **disjoint** $S, T \subseteq [n]$,

$$f(S \sqcup T) \leq f(S) + f(T).$$

XOS

f is called **XOS** if f is a point-wise maximum of additive functions. i.e. there exists a colleciton of additve functions $g_1, \cdots, g_k$ such that

$$f(S) = \max_{i=1}^{k} g_k(S) = \max_{i=1}^{k} \sum_{j \in S} g_k(\{j\}).$$

Conclusion

Implications

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This paper:

- If v_i's are XOS, and B the any (pure/mixed) BNE with no-overbidding, Price of Anarchy is tightly upper-bounded by 2.
- In case of full-information game and v_i's are submodular, we can find such BNE in polynomial time.
  - If we take the submodularity down to XOS, there's a full-information instance breaking the polynomial runtime.

Further readings

- On sub-additive v_i's, pure NE achieves PoA 2, while mixed NE attains PoA within $[2.061, 4]$.
- What if we select simultaneous FPA, instead of SPA?
- What if v_i's are **supermodular**?

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