Bayesian and Randomized Clock Auctions

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Introduction

A *clock auction* is a multi-round mechanism that suggests a personal clock price, that is increasing over rounds, to each buyer. In each round, every buyer chooses either to leave or stay.

The authors construct clock auction mechanisms,
which give $O(\log \log k)$-approximation of maximum expected welfare.

1. Deterministic single-price clock auction under full access to priors
2. Deterministic clock auction under limited access to priors
3. Randomized prior-free clock auction

Basic Notations

- buyer $i \in N := [1, n] = [n]$
- i's private value v_i, $\mathbf{v} := (v_i)_{i \in N}$
- feasibility constraint $\mathcal{F} \subseteq 2^N$
  - $\mathcal{F}$ is *downward-closed*: $F \in \mathcal{F}$ implies $F' \in \mathcal{F}$ for every $F' \subseteq F$
  - $\mathcal{S} = (S_1, \ldots, S_k)$ denotes the collection of maximal feasible sets in $\mathcal{F}$.

Bayesian Setting, Prior-free Setting

1. Bayesian Setting:
   - each value $v_i \sim D_i$, $\mathbf{v} \sim \mathbf{D} := \times_{i \in N} D_i$
   - the expected social welfare of an auction $\textsc{Auc}$, $\mathbb{E}_{\mathbf{v} \sim \mathbf{D}}[\textsc{Auc}]$
   - $\mathrm{OPT} = \mathbb{E}_{\mathbf{v} \sim \mathbf{D}}[\max_{F \in \mathcal{F}} \{\sum_{i \in F} v_i\}]$ $(= \mathbb{E}[\max_{S \in \mathcal{S}} \sum_{i \in S} v_i])$
2. Prior-free Setting:
   - $\mathbf{v}$ is chosen adversarially
   - auction is randomized

$O(\log\log k)$-Approx. with Full Prior Info.

Theorem

Every downward-closed setting with k maximal sets admits a deterministic single-price clock auction that obtains an $O(\log\log k)$ approximation to the expected optimal welfare if the full distribution $\mathbf{D}$ is known.

For the following two auctions, we choose one with the higher expected social welfare:

1. Let $p_i = 0$ for all $i \in [n]$, then choose $\arg\max_{S\in\mathcal{S}} \sum_{i\in S} \mathbb{E}[v_i]$.
2. Choose a $j \in [0, \log(10\log k + 1)]$ and let $p_i = \Delta \cdot 2^{1-j}$ for all $i \in [n]$, where

$$\Delta = \mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S} v_i \cdot \mathbb{I}\left[v_i > t_S\right] \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| \leqslant 10\log k + 1\right]\right)\right].$$

Proof for Full Prior Info.

Notations on High-Value Buyers

Note that $\mathcal{S}$ denotes the set of the maximal feasible set.
Let us define

$$S(t, \mathbf{v}) := \{i \in S \mid v_i > t\}.$$

Then, the threshold t_S is the value satisfying

$$\mathbb{E}_{\mathbf{v}}\left[|S(t_S, \mathbf{v})|\right] = \log k.$$

Then, $S(t_S, \mathbf{v})$ denotes roughly $\log k$-top high-value buyers in S.

Lemma

$$\mathbf{Pr}\left[\exists x \in [0, t_S) : |S(x, \mathbf{v})| > 10 \cdot \mathbb{E}_{\mathbf{v}}[|S(x, \mathbf{v})|]\right] = o(1/k^2).$$

Proof for Full Prior Info.

Low-High Decomposition

We divide OPT into two components, LOW and HIGH.

$$\mathrm{OPT} = \mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} v_i\right] \leqslant \underbrace{\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} \check{v}_{i,S}\right]}_{=:\mathrm{LOW}} + \underbrace{\mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} \hat{v}_{i,S}\right]}_{=:\mathrm{HIGH}},$$

where

$$\check{v}_{i,S} := \min\{t_S, v_i\} \text{ and } \hat{v}_{i,S} := v_i \cdot \mathbb{I}\left[v_i > t_S\right].$$

Proof for Full Prior Info.

High-Core and High-Tail (1)

Recall HIGH, and define HIGH-CORE.

- HIGH $= \mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} v_i \cdot \mathbb{I}\left[v_i > t_S\right]\right]$
- HIGH-CORE $:= \mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} \hat{v}_{i,S} \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| \leqslant 10 \log k + 1\right]\right]$

Then,

$$\text{HIGH} \leqslant \text{HIGH-CORE} + \mathbb{E}_{\mathbf{v}}\left[\max_{S \in \mathcal{S}}\left[\sum_{i \in S} \hat{v}_{i,S} \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| > 10 \log k + 1\right]\right]\right]$$

Proof for Full Prior Info.

High-Core and High-Tail (2)

$$\text{HIGH} \leqslant \text{HIGH-CORE} + \mathbb{E}_{\mathbf{v}}\left[\max_{S\in\mathcal{S}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|>10\log k+1\right]\right]\right]$$

$$\leqslant \text{HIGH-CORE} + \sum_{S\in\mathcal{S}}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|>10\log k+1\right]\right]$$

$$\leqslant \text{HIGH-CORE} + \sum_{S\in\mathcal{S}}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in S}v_i\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})\setminus\{i\}|>10\log k\right]\right]$$

$$\leqslant \text{HIGH-CORE} + \underbrace{\sum_{S\in\mathcal{S}}\sum_{i\in S}\mathbb{E}[v_i]\cdot\mathbf{Pr}\left[|S(t_S,\mathbf{v})\setminus\{i\}|>10\log k\right]}_{=:\text{HIGH-TAIL}}$$

Proof for Full Prior Info.

High-Core and High-Tail (2)

$$\begin{aligned}
\text{HIGH} &\leqslant \text{HIGH-CORE} + \mathbb{E}_{\mathbf{v}}\left[\max_{S\in\mathcal{S}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|>10\log k+1\right]\right]\right] \\
&\leqslant \text{HIGH-CORE} + \sum_{S\in\mathcal{S}}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|>10\log k+1\right]\right] \\
&\leqslant \text{HIGH-CORE} + \sum_{S\in\mathcal{S}}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in S}v_i\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})\setminus\{i\}|>10\log k\right]\right] \\
&\leqslant \text{HIGH-CORE} + \underbrace{\sum_{S\in\mathcal{S}}\sum_{i\in S}\mathbb{E}[v_i]\cdot\mathbf{Pr}\left[|S(t_S,\mathbf{v})\setminus\{i\}|>10\log k\right]}_{=:\text{HIGH-TAIL}}
\end{aligned}$$

Proof for Full Prior Info.

High-Core and High-Tail (2)

$$
\begin{aligned}
\mathrm{HIGH} &\leqslant \mathrm{HIGH\text{-}CORE} + \mathbb{E}_{\mathbf{v}}\left[\max_{S\in\mathcal{S}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})| > 10\log k + 1\right]\right]\right] \\
&\leqslant \mathrm{HIGH\text{-}CORE} + \sum_{S\in\mathcal{S}}\mathbb{E}_{\mathbf{v}}\left[\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})| > 10\log k + 1\right]\right] \\
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&\leqslant \mathrm{HIGH\text{-}CORE} + \underbrace{\sum_{S\in\mathcal{S}}\sum_{i\in S}\mathbb{E}[v_i]\cdot\mathbf{Pr}\left[|S(t_S,\mathbf{v})\setminus\{i\}| > 10\log k\right]}_{=:\mathrm{HIGH\text{-}TAIL}}
\end{aligned}
$$

Proof for Full Prior Info.

High-Core and High-Tail (2)

$$
\begin{aligned}
\text{HIGH} &\leqslant \text{HIGH-CORE} + \mathbb{E}_{\mathbf{v}}\left[\max_{S \in \mathcal{S}}\left[\sum_{i \in S} \hat{v}_{i,S} \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| > 10 \log k + 1\right]\right]\right] \\
&\leqslant \text{HIGH-CORE} + \sum_{S \in \mathcal{S}} \mathbb{E}_{\mathbf{v}}\left[\sum_{i \in S} \hat{v}_{i,S} \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| > 10 \log k + 1\right]\right] \\
&\leqslant \text{HIGH-CORE} + \sum_{S \in \mathcal{S}} \mathbb{E}_{\mathbf{v}}\left[\sum_{i \in S} v_i \cdot \mathbb{I}\left[|S(t_S, \mathbf{v}) \setminus \{i\}| > 10 \log k\right]\right] \\
&\leqslant \text{HIGH-CORE} + \underbrace{\sum_{S \in \mathcal{S}} \sum_{i \in S} \mathbb{E}[v_i] \cdot \mathbf{Pr}\left[|S(t_S, \mathbf{v}) \setminus \{i\}| > 10 \log k\right]}_{=:\text{HIGH-TAIL}}
\end{aligned}
$$

Proof for Full Prior Info.

Cover High-Tail

Lemma: $\mathbf{Pr}\left[\exists x \in [0, t_S) : |S(x, \mathbf{v})| > 10 \cdot \mathbb{E}_{\mathbf{v}}[|S(x, \mathbf{v})|]\right] = o(1/k^2)$.

Then,

$$\begin{aligned} \text{HIGH-TAIL} &= \sum_{S \in \mathcal{S}} \sum_{i \in S} \mathbb{E}[v_i] \cdot \mathbf{Pr}\left[|S(t_S, \mathbf{v}) \setminus \{i\}| > 10 \log k\right] \\ &\leqslant \sum_{S \in \mathcal{S}} \sum_{i \in S} \mathbb{E}[v_i] \cdot o(1/k^2). \end{aligned}$$

Note that the first auction, letting $p_i = 0$ for all $i \in [n]$, will get the expected social welfare of

$$\max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{i \in S} v_i\right] \geqslant \frac{1}{k} \cdot \sum_{S \in \mathcal{S}} \sum_{i \in S} \mathbb{E}[v_i] \geqslant \sum_{S \in \mathcal{S}} \sum_{i \in S} \mathbb{E}[v_i] \cdot o(1/k^2).$$

Proof for Full Prior Info.

Low-Core and Low-Tail

Recall LOW, and define LOW-CORE.

- LOW $= \mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} \min(v_i, t_S)\right]$
- LOW-CORE :=

$$\mathbb{E}\left[\max_{S \in \mathcal{S}} \check{v}_{i,S} \cdot \mathbb{I}\left[\forall x \in [0, t_S) : |S(x, \mathbf{v})| \leqslant 10 \cdot \mathbb{E}_{\mathbf{v}}[|S(x, \mathbf{v})|] + 1\right]\right]$$

We omit details. Similar to the case of HIGH-TAIL,

$$\text{LOW} \leqslant \text{LOW-CORE} + \text{LOW-TAIL}$$

and LOW-TAIL is covered by the "zero-price auction".

Proof for Full Prior Info.

Cover Low-Core

1. Note that $\sum_{i \in S} \check{v}_{i,S} = \int_0^{t_S} |S(x, \mathbf{v})| dx$.
2. $\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) := \mathbb{I}\left[\forall x \in [0, t_S) : |S(x, \mathbf{v})| \leqslant 10 \cdot \mathbb{E}_{\mathbf{v}}[|S(x, \mathbf{v})|] + 1\right].$

$$\text{LOW-CORE} = \mathbb{E}\left[\max_{S \in \mathcal{S}} \left[\check{v}_{i,S} \cdot \mathbb{I}_{\mathcal{E}(S)}(\mathbf{v})\right]\right]$$

$$\begin{aligned}
\text{LOW-CORE} &= \mathbb{E}\left[\max_{S \in \mathcal{S}} \left[\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) \cdot \int_0^{t_S} |S(x, \mathbf{v})| dx\right]\right] \\
&\leqslant \max_{S \in \mathcal{S}} \left[\int_0^{t_S} (10 \cdot \mathbb{E}(|S(x, \mathbf{v})| + 1) dx\right] \\
&\leqslant \max_{S \in \mathcal{S}} \left[\int_0^{t_S} (11 \cdot \mathbb{E}(|S(x, \mathbf{v})|) dx\right] \\
&\leqslant 11 \cdot \max_{S \in \mathcal{S}} \mathbb{E}\left[\int_0^{t_S} (|S(x, \mathbf{v})|) dx\right] \leqslant 11 \cdot \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{i \in S} v_i\right]
\end{aligned}$$

$\Longrightarrow$ LOW-CORE is also covered by the zero-price auction.

Proof for Full Prior Info.

Cover Low-Core

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$$\text{LOW-CORE} = \mathbb{E}\left[\max_{S \in \mathcal{S}} \left[\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) \cdot \int_0^{t_S} |S(x, \mathbf{v})| dx\right]\right]$$

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\end{aligned}$$

$\implies$ LOW-CORE is also covered by the zero-price auction.

Proof for Full Prior Info.

Cover Low-Core

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$$\text{LOW-CORE} = \mathbb{E}\left[\max_{S \in \mathcal{S}}\left[\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) \cdot \int_0^{t_S} |S(x, \mathbf{v})| dx\right]\right]$$
$$\leqslant \max_{S \in \mathcal{S}}\left[\int_0^{t_S} (10 \cdot \mathbb{E}(|S(x, \mathbf{v})| + 1) dx\right]$$
$$\leqslant \max_{S \in \mathcal{S}}\left[\int_0^{t_S} (11 \cdot \mathbb{E}(|S(x, \mathbf{v})|) dx\right]$$
$$\leqslant 11 \cdot \max_{S \in \mathcal{S}} \mathbb{E}\left[\int_0^{t_S} (|S(x, \mathbf{v})|) dx\right] \leqslant 11 \cdot \max_{S \in \mathcal{S}} \mathbb{E}\left[\sum_{i \in S} v_i\right]$$

$\implies$ LOW-CORE is also covered by the zero-price auction.

Proof for Full Prior Info.

Cover Low-Core

1. Note that $\sum_{i \in S} \check{v}_{i,S} = \int_0^{t_S} |S(x, \mathbf{v})| dx$.
2. $\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) := \mathbb{I}\left[\forall x \in [0, t_S) : |S(x, \mathbf{v})| \leqslant 10 \cdot \mathbb{E}_{\mathbf{v}}[|S(x, \mathbf{v})|] + 1\right].$

$$
\begin{aligned}
\text{LOW-CORE} &= \mathbb{E}\left[\max_{S \in \mathcal{S}} \left[\mathbb{I}_{\mathcal{E}(S)}(\mathbf{v}) \cdot \int_0^{t_S} |S(x, \mathbf{v})| dx\right]\right] \\
&\leqslant \max_{S \in \mathcal{S}} \left[\int_0^{t_S} (10 \cdot \mathbb{E}(|S(x, \mathbf{v})| + 1) dx\right] \\
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\end{aligned}
$$

$\implies$ LOW-CORE is also covered by the zero-price auction.

Proof for Full Prior Info.

High-Core (1)

Lemma

Let m be any positive integer, and let

$$\Delta := \mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|\leqslant m\right]\right)\right].$$

There exists a uniform price p such that

$$\Delta \leqslant O(\log m)\cdot\mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S}\mathbb{E}[v_i \mid v_i\geqslant p]\cdot\mathbb{I}\left[v_i\geqslant p\right]\right)\right].$$

Moreover, $\mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S}\mathbb{E}[v_i \mid v_i\geqslant p]\cdot\mathbb{I}\left[v_i\geqslant p\right]\right)\right]$
is the expected welfare of price-p clock auction.

Proof for Full Prior Info.

High-Core (2)

Recall the second auction,

- choose a $j \in [0, \log(10 \log k + 1)]$ and
- let $p_i = \Delta \cdot 2^{1-j}$ for all $i \in [n]$, where
- Δ is chosen by letting $m = 10 \log k + 1$.

Then,

$$\text{HIGH-CORE} = \mathbb{E}\left[\max_{S \in \mathcal{S}} \sum_{i \in S} \hat{v}_{i,S} \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| \leqslant 10 \log k + 1\right]\right]$$
$$= \Delta \leqslant O(\log \log k) \cdot \mathbb{E}[\textsc{Auc}]$$

$O(\log\log k)$-Approx. with Full Prior Info.

Recall

Theorem

Every downward-closed setting with k maximal sets admits a deterministic single-price clock auction that obtains an $O(\log\log k)$ approximation to the expected optimal welfare if the full distribution $\mathbf{D}$ is known.

For the following two auctions, we choose one with the higher expected social welfare:

1. Let $p_i = 0$ for all $i \in [n]$, then choose $\arg\max_{S\in\mathcal{S}} \sum_{i\in S} \mathbb{E}[v_i]$.
2. Choose a $j \in [0, \log(10\log k + 1)]$ and let $p_i = \Delta \cdot 2^{1-j}$ for all $i \in [n]$, where

$$\Delta = \mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S} v_i \cdot \mathbb{I}\left[v_i > t_S\right] \cdot \mathbb{I}\left[|S(t_S, \mathbf{v})| \leqslant 10\log k + 1\right]\right)\right].$$

$O(\log\log k)$-Approx. with Limited Prior Info.

Theorem

Every downward-closed setting with k maximal sets admits a uniform-price deterministic clock auction that obtains an $O(\log\log k)$ approximation to the expected optimal welfare if $\mathbb{E}[v_i]$ for each $i \in N$ and $OPT = \mathbb{E}_{\mathbf{v}}[\max_{S\in\mathcal{S}} \sum_{i\in S} v_i]$ is known.

1. If $\max_{S\in\mathcal{S}} \sum_{i\in S} \mathbb{E}[v_i] \geqslant \mathrm{OPT}/\log\log k$, then choose $\arg\max_{S\in\mathcal{S}} \sum_{i\in S} \mathbb{E}[v_i]$.
2. Otherwise, increase p until
   - the active buyers A is feasible, or
   - there exists some feasible $F \subseteq A$ and $|F| \cdot p \geqslant g$.

$O(\log\log k)$-Approx. with Limited Prior Info.

Desired goal g

Lemma

Let m be any positive integer, and let

$$\Delta := \mathbb{E}\left[\max_{S\in\mathcal{S}}\left(\sum_{i\in S}\hat{v}_{i,S}\cdot\mathbb{I}\left[|S(t_S,\mathbf{v})|\leqslant m\right]\right)\right].$$

There exists an $\alpha = O(\log m)$ and a uniform price p such that

$$\Delta \leqslant \alpha\cdot p\cdot\mathbb{E}\left[\max_{S\in\mathcal{S}}\sum_{i\in S}\mathbb{I}\left[v_i\geqslant p\right]\right] \text{ or } \Delta \leqslant \alpha\cdot\mathbb{E}\left[\max_{i\in N}\mathbb{E}[v\mid v_i\geqslant p]\cdot\mathbb{I}\left[v_i\geqslant p\right]\right]$$

Let the goal $g = \mathrm{OPT}/4\alpha$.

Proof for Limited Prior Info.

Sketch of Proof

With a similar argument with the full prior info,
the first auction covers LOW + HIGH-TAIL.

Note that the second auction with price p.

- The set of active buyers A is feasible, or
- there exists some feasible $F \subseteq A$ and $|F| \cdot p \geqslant g = \mathrm{OPT}/4\alpha$.

Let $T(\mathbf{v})$ be the largest feasible set of buyers with price p.

1. When $\mathbb{E}[|T(\mathbf{v})|] \geqslant 8$, with constant probability we serve at least half the expected number of buyers and obtain the $O(\log\log k)$-approximation from the revenue.
2. When $\mathbb{E}[|T(\mathbf{v})|] < 8$, the expected social welfare is within a constant factor of the single highest value buyer.

$O(\log \log k)$-Approx. without Prior Info.

Theorem

Every downward-closed setting with k maximal sets admits a randomized clock auction that obtains an $O(\log \log k)$-approximation to the expected optimal welfare.

Hedging Auction. With probability 1/2 run of the following auctions:

1. Water-filling clock auction
2. Sampling clock auction

$O(\log\log k)$-Approx. without Prior Info.

Water-Filling Clock Auction

Water-Filling Clock Auction. Let $A \leftarrow N$ and $p_i \leftarrow 0$ for all $i \in N$, and do the following steps while A becomes feasible.

1. Let $W \leftarrow \arg\max_{\text{feasible } S \subseteq A} \left[\sum_{i \in S} p_i\right]$.
2. Let $\ell \leftarrow \min_{i \in A \setminus W} \{p_i\}$.
3. For each buyer $i \in A \setminus W$ with $p_i = \ell$,
   3.1 Increase p_i.
   3.2 If i rejects, $A \leftarrow A \setminus \{i\}$.

The above auction obtains welfare at least

$$\max_{p \in P, F \in \mathcal{F}} \left[p \cdot |i \in F : v_i \geqslant p|\right] / 2.$$

This then translates to an $O(\log |S|)$-approx. for any given set S.

$O(\log \log k)$-Approx. without Prior Info.

Sampling clock auction

Sampling Clock Auction.

1. For each buyer i, "sample" the value of i with probability $1/2$.
2. Let $R \leftarrow \arg\max_{S \in \mathcal{S}} v(S \cap T)$. Return $R \backslash T$.

$O(\log \log k)$-Approx. without Prior Info.

Recall

Theorem

Every downward-closed setting with k maximal sets admits a randomized clock auction that obtains an $O(\log \log k)$-approximation to the expected optimal welfare.

Hedging Auction. With probability $1/2$ run of the following auctions:

1. Water-filling clock auction
2. Sampling clock auction

Proof for Full Prior Info.

When Sets of High Value are Small

Let

1. S_{top} denote the $60 \log k$ highest value buyers in S, and
2. $\tau(S)$ denote the threshold ($=$ the smallest value in S_{top}).

Then, the water-filling clock auction gives
$O(\log \log k)$-approximation to $v(O_{\text{top}})$ where O is an optimal feasible set.

Remains to show that the sampling clock auction obtains an
$O(\log \log k)$-approximation to $v(O) - v(O_{\text{top}})$.

Proof for Full Prior Info.

When Sets of High Value are Large (1)

We use the following lemma.

Lemma

When running the sampling auction (T denotes sampled buyers),

$$\mathbf{Pr}\left[\exists x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \notin [1/9, 8/9] \cdot |S(x, \mathbf{v})|\right] = o(1/k^2).$$

Then,

$$\mathbf{Pr}\left[\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|\right] \\ = 1 - o(1/k^2).$$

Proof for No Prior Info.

When Sets of High Value are Large (2)

Recall the water-falling clock auction obtains welfare at least

$$\max_{p \in P, F \in \mathcal{F}} \left[p \cdot |i \in F : v_i \geqslant p| \right] / 2.$$

The auction covers $\mathrm{OPT} = v(O) < 100 \cdot \max_{S \in \mathcal{S}} v(S_{\mathrm{top}})$.

Thus we can assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\mathrm{top}})$.

Let $S^* := \arg\max_{S \in \mathcal{S}} v(S \cap T)$.
Then the random sampling auction obtains welfare $v(S^* \cap U)$ where $U = N \backslash T$.

Proof for No Prior Info.

When Sets of High Value are Large (3)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
2. $S^* = \arg\max_{S \in \mathcal{S}} v(S \cap T)$.
3. The random sampling auction obtains $v(S^* \cap U)$.
4. With probability $1 - o(1/k)$,

$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

With probability $1 - o(1/k)$,

$$\begin{aligned} v(S^* \cap T) &\geqslant v(O \cap T) \geqslant \int_0^{\tau(O)} |T \cap O(x, \mathbf{v})| dx \\ &\geqslant \int_0^{\tau(O)} \frac{1}{9} \cdot |O(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(O) - v(O_{\text{top}})) \\ &\geqslant \frac{11 v(O)}{100} \end{aligned}$$

Proof for No Prior Info.

When Sets of High Value are Large (3)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
2. $S^* = \arg\max_{S \in \mathcal{S}} v(S \cap T)$.
3. The random sampling auction obtains $v(S^* \cap U)$.
4. With probability $1 - o(1/k)$,

$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

With probability $1 - o(1/k)$,

$$v(S^* \cap T) \overset{2}{\geqslant} v(O \cap T) \geqslant \int_0^{\tau(O)} |T \cap O(x, \mathbf{v})| dx$$

$$\geqslant \int_0^{\tau(O)} \frac{1}{9} \cdot |O(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(O) - v(O_{\text{top}}))$$

$$\geqslant \frac{11 v(O)}{100}$$

Proof for No Prior Info.

When Sets of High Value are Large (3)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
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With probability $1 - o(1/k)$,

$$v(S^* \cap T) \geqslant v(O \cap T) \geqslant \int_0^{\tau(O)} |T \cap O(x, \mathbf{v})| dx$$

$$\geqslant \int_0^{\tau(O)} \frac{1}{9} \cdot |O(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(O) - v(O_{\text{top}}))$$

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Proof for No Prior Info.

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$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

With probability $1 - o(1/k)$,

$$v(S^* \cap T) \geqslant v(O \cap T) \geqslant \int_0^{\tau(O)} |T \cap O(x, \mathbf{v})| dx$$

$$\overset{4}{\geqslant} \int_0^{\tau(O)} \frac{1}{9} \cdot |O(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(O) - v(O_{\text{top}}))$$

$$\geqslant \frac{11 v(O)}{100}$$

Proof for No Prior Info.

When Sets of High Value are Large (3)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
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With probability $1 - o(1/k)$,

$$\begin{aligned} v(S^* \cap T) \geqslant v(O \cap T) &\geqslant \int_0^{\tau(O)} |T \cap O(x, \mathbf{v})| dx \\ &\geqslant \int_0^{\tau(O)} \frac{1}{9} \cdot |O(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(O) - v(O_{\text{top}})) \\ &\stackrel{1}{\geqslant} \frac{11 v(O)}{100} \end{aligned}$$

Proof for No Prior Info.

When Sets of High Value are Large (4)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
2. $S^* = \arg\max_{S \in \mathcal{S}} v(S \cap T)$.
3. The random sampling auction obtains $v(S^* \cap U)$.
4. With probability $1 - o(1/k)$, $v(S^* \cap T) \geqslant \frac{11v(O)}{100}$ and

$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

Again, with probability $1 - o(1/k)$,

$$v(S^* \cap U) \geqslant \int_0^{\tau(S^*)} |U \cap S^*(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(S^*) - v(S^*_{\text{top}}))$$

$$\geqslant \frac{1}{9} \cdot \left(v(S^*) - \frac{v(O)}{100} \right) \geqslant \frac{1}{9} \cdot \left(v(S^* \cap T) - \frac{v(O)}{100} \right)$$

$$\geqslant \frac{1}{9} \cdot \left(\frac{11v(O)}{100} - \frac{v(O)}{100} \right) \geqslant \frac{v(O)}{900}$$

Proof for No Prior Info.

When Sets of High Value are Large (4)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
2. $S^* = \arg\max_{S \in \mathcal{S}} v(S \cap T)$.
3. The random sampling auction obtains $v(S^* \cap U)$.
4. With probability $1 - o(1/k)$, $v(S^* \cap T) \geqslant \frac{11 v(O)}{100}$ and

$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

Again, with probability $1 - o(1/k)$,

$$v(S^* \cap U) \geqslant \int_0^{\tau(S^*)} |U \cap S^*(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(S^*) - v(S^*_{\text{top}}))$$

$$\overset{1}{\geqslant} \frac{1}{9} \cdot \left(v(S^*) - \frac{v(O)}{100} \right) \geqslant \frac{1}{9} \cdot \left(v(S^* \cap T) - \frac{v(O)}{100} \right)$$

$$\geqslant \frac{1}{9} \cdot \left(\frac{11 v(O)}{100} - \frac{v(O)}{100} \right) \geqslant \frac{v(O)}{900}$$

Proof for No Prior Info.

When Sets of High Value are Large (4)

1. We assume $v(O)/100 \geqslant \max_{S \in \mathcal{S}} v(S_{\text{top}})$.
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3. The random sampling auction obtains $v(S^* \cap U)$.
4. With probability $1 - o(1/k)$, $v(S^* \cap T) \geqslant \frac{11v(O)}{100}$ and

$$\forall S \in \mathcal{S}, \forall x \in [0, \tau(S)] : |T \cap S(x, \mathbf{v})| \in [1/9, 8/9] \cdot |S(x, \mathbf{v})|$$

Again, with probability $1 - o(1/k)$,

$$v(S^* \cap U) \geqslant \int_0^{\tau(S^*)} |U \cap S^*(x, \mathbf{v})| dx \geqslant \frac{1}{9} \cdot (v(S^*) - v(S^*_{\text{top}}))$$
$$\geqslant \frac{1}{9} \cdot \left(v(S^*) - \frac{v(O)}{100} \right) \geqslant \frac{1}{9} \cdot \left(v(S^* \cap T) - \frac{v(O)}{100} \right)$$
$$\overset{4}{\geqslant} \frac{1}{9} \cdot \left(\frac{11v(O)}{100} - \frac{v(O)}{100} \right) \geqslant \frac{v(O)}{900}$$

Thank you