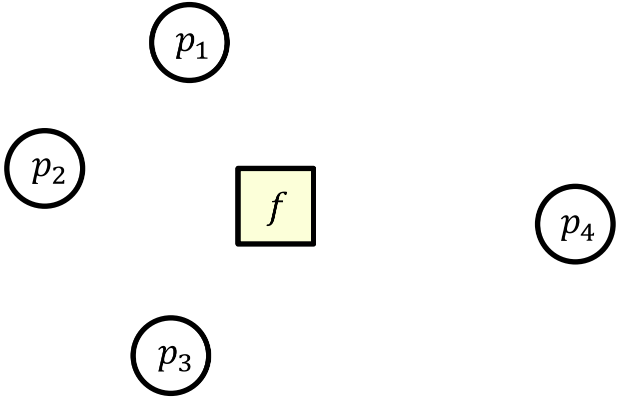
Learning-Augmented Mechanism Design

연세대학교 이국렬

Single Facility Location

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$

Goal : choose a location $f \in \mathbb{R}^2$ for a facility to minimize the social cost

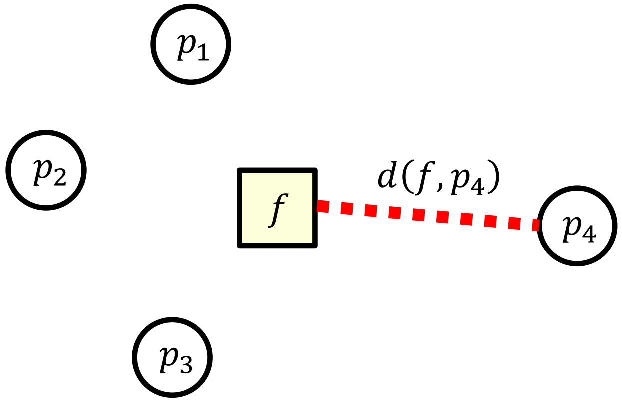


Single Facility Location

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$

Goal : choose a location $f \in \mathbb{R}^2$ for a facility to minimize the social cost

- Each agent i suffers a cost $d(f, p_i)$



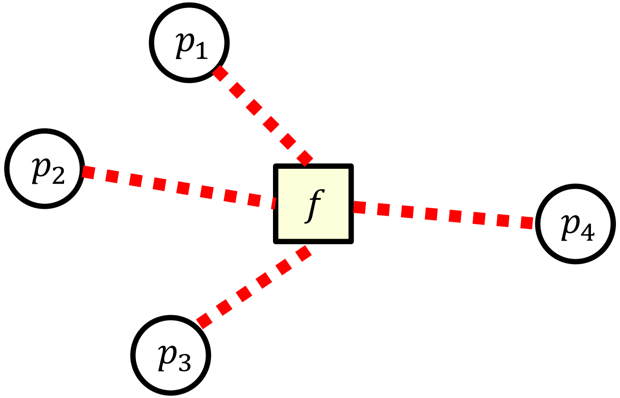
Single Facility Location

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$

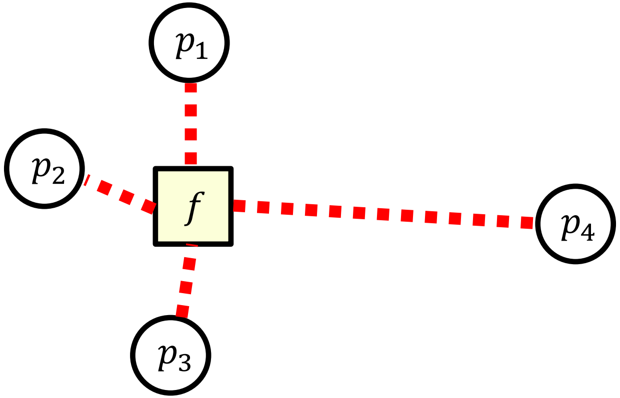
Goal : choose a location $f \in \mathbb{R}^2$ for a facility to minimize the social cost

- Each agent i suffers a cost $d(f, p_i)$

- Egalitarian social cost : $\max_{p \in P} d(f, p)$, Utilitarian social cost : $\sum_{p \in P} d(f, p) / n$



Egalitarian

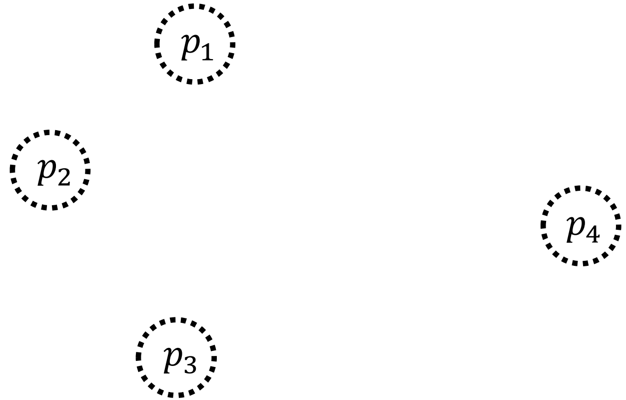


Utilitarian

Strategic model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

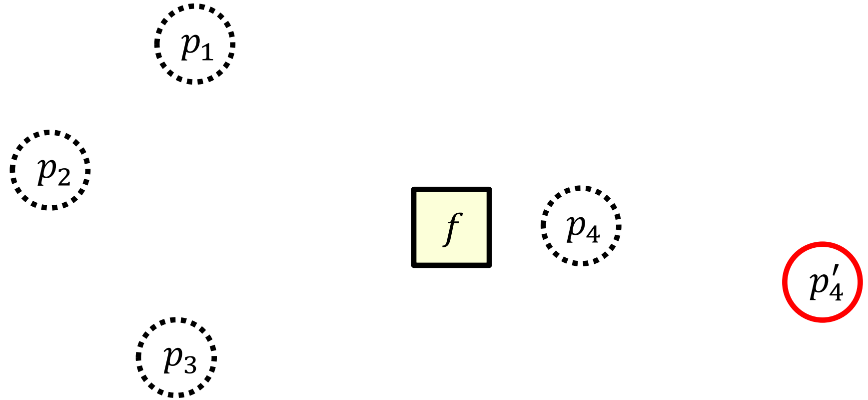
- A mechanism needs to ask the agents to report their preferred locations



Strategic model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

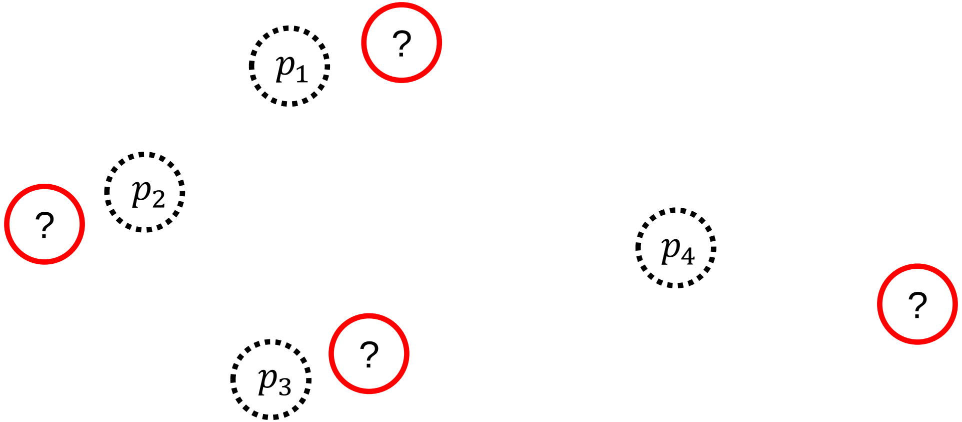
- A mechanism needs to ask the agents to report their preferred locations
- Each agent wants to reduce $d(f, p_i)$ → misreport his preferred location



Strategic model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

- A mechanism needs to ask the agents to report their preferred locations
- Each agent wants to reduce $d(f, p_i)$ → misreport his preferred location
- The identity of the agents is not distinguishable



Strategic model

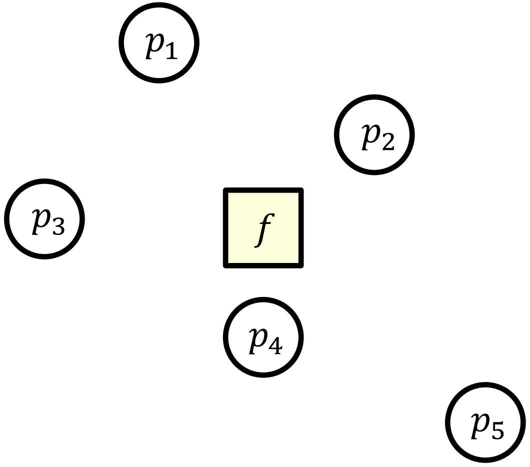
Mechanism $f: \mathbb{R}^{2n} \to \mathbb{R}^2$ is **strategyproof** if

- truthful reporting is a dominant strategy for every agent
  - For all $i \in [n]$, $d\big(p_i, f(P)\big) \leq d\big(p_i, f(P_{-i}, p_i')\big)$ for all $p_i' \in \mathbb{R}^2$

Coordinatewise Median Mechanism

CM mechanism takes as input the locations $P = \{(x_i, y_i)\}_{i \in [n]}$

- The x-coordinate of the facility : the median of $\{x_i\}_{i \in [n]}$
- The y-coordinate of the facility : the median of $\{y_i\}_{i \in [n]}$
- n is even → the smaller of the two medians is returned.



Coordinatewise Median Mechanism

CM mechanism takes as input the locations $P = \{(x_i, y_i)\}_{i\in[n]}$

- The x-coordinate of the facility : the median of $\{x_i\}_{i\in[n]}$
- The y-coordinate of the facility : the median of $\{y_i\}_{i\in[n]}$
- n is even → the smaller of the two medians is returned.

CM mechanism : Deterministic & Strategyproof

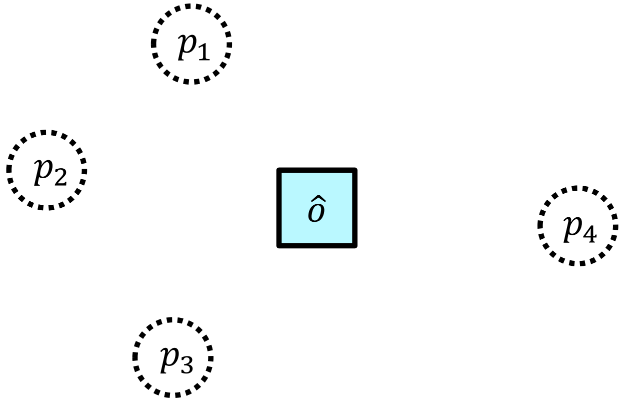
Egalitarian social cost → 2-approximation (best)

Utilitarian social cost → $\sqrt{2}$-approximation (best)

Prediction model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

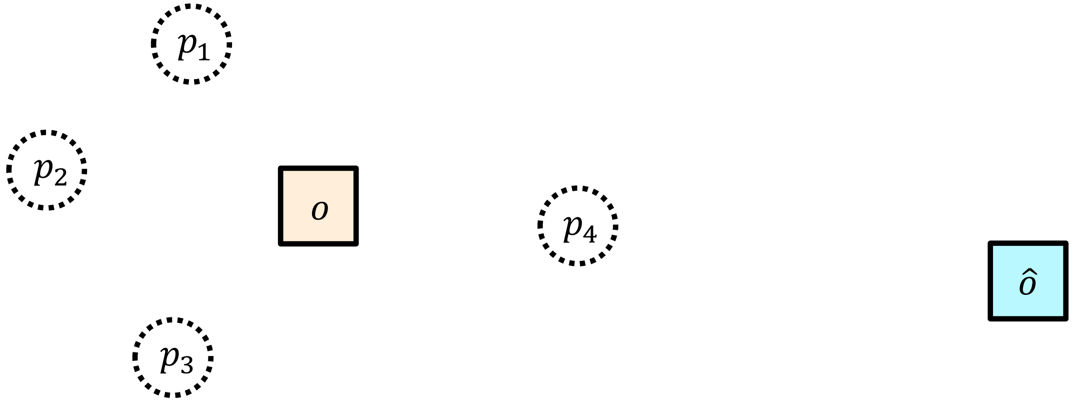
$\hat{o}$: a prediction regarding the optimal facility location $o(P)$



Prediction model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

$\hat{o}$: a prediction regarding the optimal facility location $o(P)$ (may be wrong)



Prediction model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

$\hat{o}$: a prediction regarding the optimal facility location $o(P)$ (may be wrong)

$f(P, \hat{o})$: output solution of a mechanism f

C : social cost function (ex. egalitarian, utilitarian)

Prediction model

A mechanism f is **$\boldsymbol{\alpha}$-consistent** if it achieves α-approximation when $\hat{\boldsymbol{o}} = \boldsymbol{o}(\boldsymbol{P})$.

$$\max_{P} \left\{ \frac{C\big(f\big(P, o(P)\big), P\big)}{C(o(P), P)} \right\} \leq \alpha$$

A mechanism f is **$\boldsymbol{\beta}$-robust** if it achieves β-approximation when $\hat{\boldsymbol{o}}$ **can be arbitrarily wrong**.

$$\max_{P, \hat{o}} \left\{ \frac{C(f(P, \hat{o}), P)}{C(o(P), P)} \right\} \leq \beta$$

Prediction model

η : the prediction error

$$\eta(\hat{o}, P) = \frac{d\big(\hat{o}, o(P)\big)}{C(o(P), P)}$$

A mechanism f achieves a $\gamma(\eta)$-approximation if

$$\max_{P,\hat{o}\,:\,\eta(\hat{o},P)\leq\eta} \left\{ \frac{C(f(P,\hat{o}), P)}{C(o(P), P)} \right\} \leq \gamma(\eta)$$

→ $\gamma(0)$-consistent & $\gamma(\infty)$-robust mechanism

Prediction model

$P = \{p_1, \cdots, p_n\}$: a set of preferred locations for the n agents in $\mathbb{R}^2$ (unknown)

$\hat{o}$: a prediction regarding the optimal facility location $o(P)$ (may be wrong)

$f(P, \hat{o})$: output solution of a mechanism f

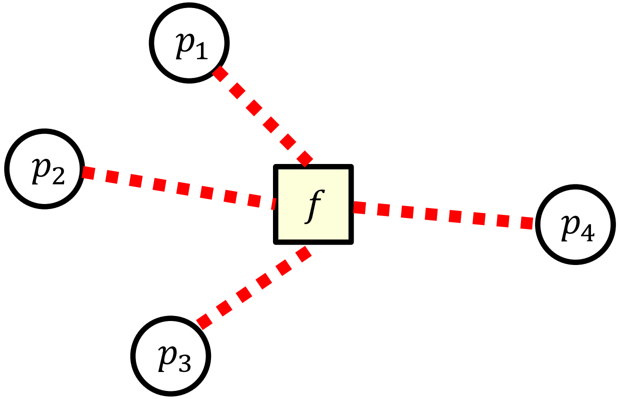
C : social cost function

Goal : Find a deterministic, strategyproof, consistent, and robust mechanism.

- with egalitarian social cost
- with utilitarian social cost

Minimizing the egalitarian social cost

Egalitarian social cost : $C^e(f,P) \coloneqq \max_{p \in P} d(f,p)$



Egalitarian

Minimizing the egalitarian social cost (1D)

Mechanism 1: MinMaxP mechanism for egalitarian social cost in one dimension.

Input: points $(p_1, \ldots, p_n) \in \mathbb{R}^n$, prediction $\hat{o} \in \mathbb{R}$
if $\hat{o} \in [\min_i p_i, \max_i p_i]$ **then**
  return $\hat{o}$
else if $\hat{o} < \min_i p_i$ **then**
  return $\min_i p_i$
else
  return $\max_i p_i$

Minimizing the egalitarian social cost (1D)

Mechanism 1: MinMaxP mechanism for egalitarian social cost in one dimension.

Input: points $(p_1, \ldots, p_n) \in \mathbb{R}^n$, prediction $\hat{o} \in \mathbb{R}$

if $\hat{o} \in [\min_i p_i, \max_i p_i]$ **then**

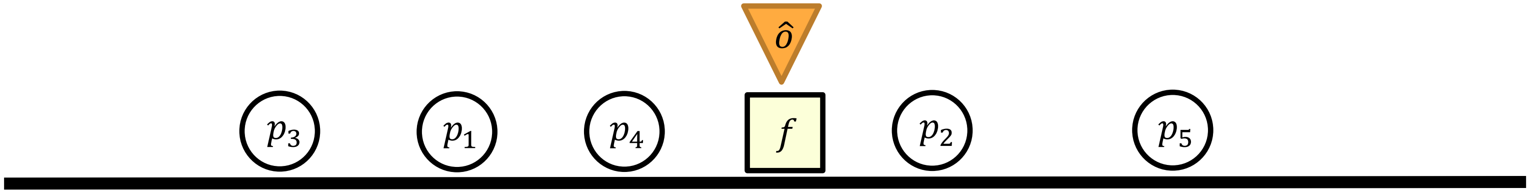
  return $\hat{o}$

else if $\hat{o} < \min_i p_i$ **then**

  return $\min_i p_i$

else

  return $\max_i p_i$



Minimizing the egalitarian social cost (1D)

Mechanism 1: MinMaxP mechanism for egalitarian social cost in one dimension.

Input: points $(p_1, \ldots, p_n) \in \mathbb{R}^n$, prediction $\hat{o} \in \mathbb{R}$

if $\hat{o} \in [\min_i p_i, \max_i p_i]$ **then**
  return $\hat{o}$
else if $\hat{o} < \min_i p_i$ **then**
  return $\min_i p_i$
else
  return $\max_i p_i$

Minimizing the egalitarian social cost (1D)

Mechanism 1: MinMaxP mechanism for egalitarian social cost in one dimension.

Input: points $(p_1, \ldots, p_n) \in \mathbb{R}^n$, prediction $\hat{o} \in \mathbb{R}$

if $\hat{o} \in [\min_i p_i, \max_i p_i]$ **then**
  return $\hat{o}$
else if $\hat{o} < \min_i p_i$ **then**
  return $\min_i p_i$
else
  return $\max_i p_i$

Minimizing the egalitarian social cost (1D)

Mechanism 1: MinMaxP mechanism for egalitarian social cost in one dimension.

Input: points $(p_1, \ldots, p_n) \in \mathbb{R}^n$, prediction $\hat{o} \in \mathbb{R}$
if $\hat{o} \in [\min_i p_i, \max_i p_i]$ **then**
  return $\hat{o}$
else if $\hat{o} < \min_i p_i$ **then**
  return $\min_i p_i$
else
  return $\max_i p_i$

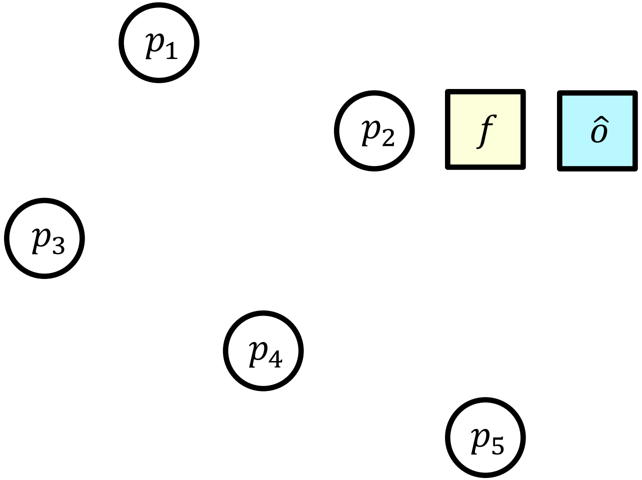
MinMaxP mechanism

- 1-consistent & 2-robust
- Strategyproof

Minimizing the egalitarian social cost (2D)

Mechanism 2: MINIMUM BOUNDING BOX mechanism for egalitarian social cost in two dimensions.

Input: points $((x_1, y_1), \ldots, (x_n, y_n)) \in \mathbb{R}^{2n}$, prediction $(x_{\hat{o}}, y_{\hat{o}}) \in \mathbb{R}^2$
$x_f = \text{MINMAXP}((x_1, \ldots, x_n), x_{\hat{o}})$
$y_f = \text{MINMAXP}((y_1, \ldots, y_n), y_{\hat{o}})$
return (x_f, y_f)



Minimizing the egalitarian social cost (2D)

Mechanism 2: MINIMUM BOUNDING BOX mechanism for egalitarian social cost in two dimensions.

Input: points $((x_1, y_1), \ldots, (x_n, y_n)) \in \mathbb{R}^{2n}$, prediction $(x_{\hat{o}}, y_{\hat{o}}) \in \mathbb{R}^2$

$x_f = \text{MINMAXP}((x_1, \ldots, x_n), x_{\hat{o}})$

$y_f = \text{MINMAXP}((y_1, \ldots, y_n), y_{\hat{o}})$

return (x_f, y_f)

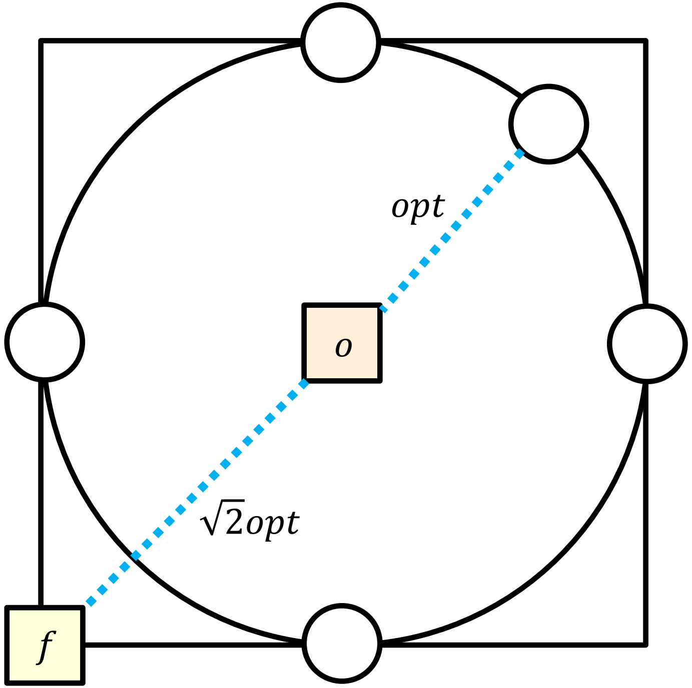
Minimum Bounding Box Mechanism

- 1-consistent & $(1+\sqrt{2})$-robust
- Strategyproof

Minimizing the egalitarian social cost (2D)

Minimum Bounding Box Mechanism

- $(1+\sqrt{2})$-robust



Minimizing the egalitarian social cost (2D)

CM Mechanism : 2-approximation → 2-consistent & 2-robust

Minimum Bounding Box Mechanism : 1-consistent & ($1+\sqrt{2}$ = 2.414)-robust

Question : Is there exists any middle-ground between these two results?

= (less than 2)-consistent & (less than $1+\sqrt{2}$)-robust?

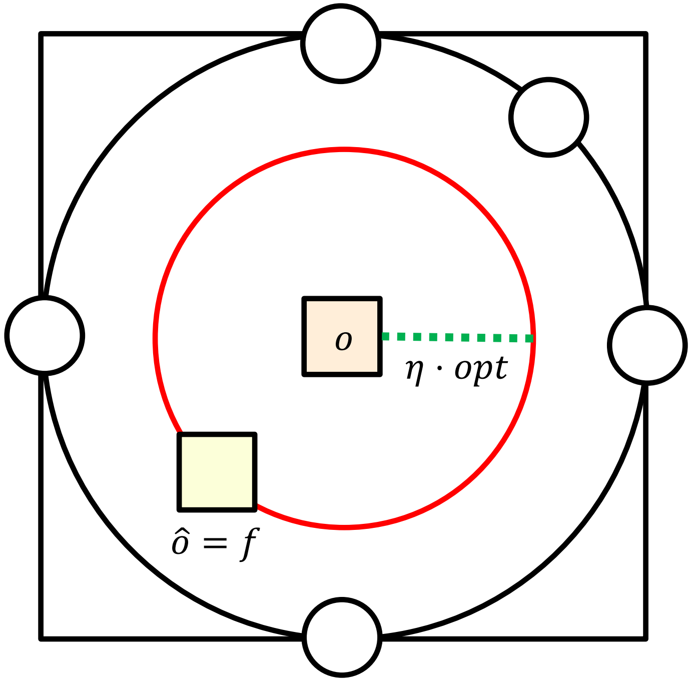
Minimizing the egalitarian social cost (2D)

Theorem : There is no deterministic and strategyproof mechanism that is $(2-\epsilon)$-consistent and $(1+\sqrt{2}-\epsilon)$-robust with respect to the egalitarian objective for any $\epsilon > 0$.

Minimizing the egalitarian social cost (2D)

Minimum Bounding Box Mechanism with prediction error η

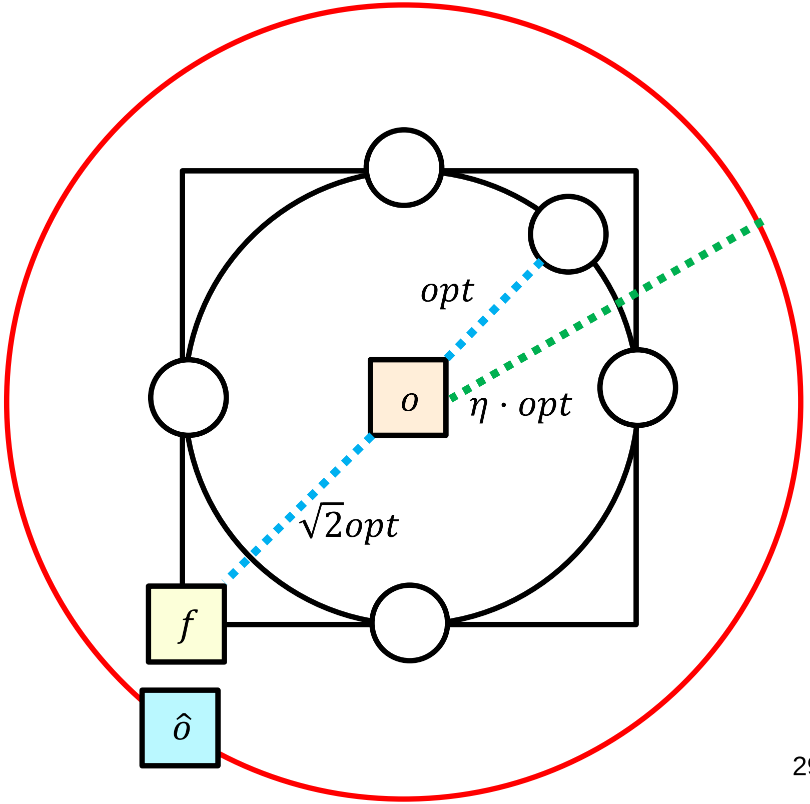
- $\min\{1+\eta, 1+\sqrt{2}\}$-approximation



Minimizing the egalitarian social cost (2D)

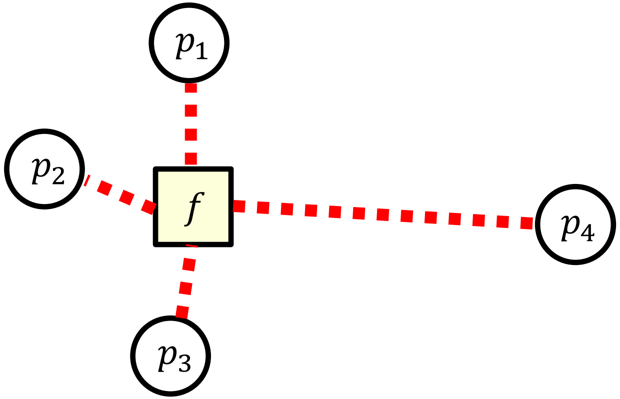
Minimum Bounding Box Mechanism with prediction error η

- $\min\{1+\eta, 1+\sqrt{2}\}$-approximation



Minimizing the utilitarian social cost

Utilitarian social cost : $C^u(f, P) \coloneqq \sum_{p \in P} d(f, p)/n$



Utilitarian

Minimizing the utilitarian social cost

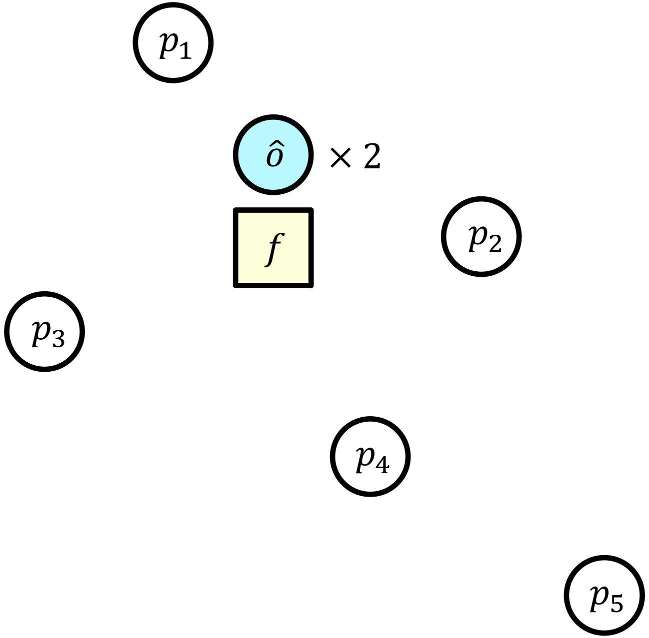
Coordinatewise Median with Predictions (CMP) mechanism

- c : confidence value in $[0,1)$
- P' : a multiset containing cn copies
- output $f(P, \hat{o}, c)$: $CM(P \cup P')$

Minimizing the utilitarian social cost

$n = 5,\ c = 0.4$

$\rightarrow cn = 2$



Minimizing the utilitarian social cost

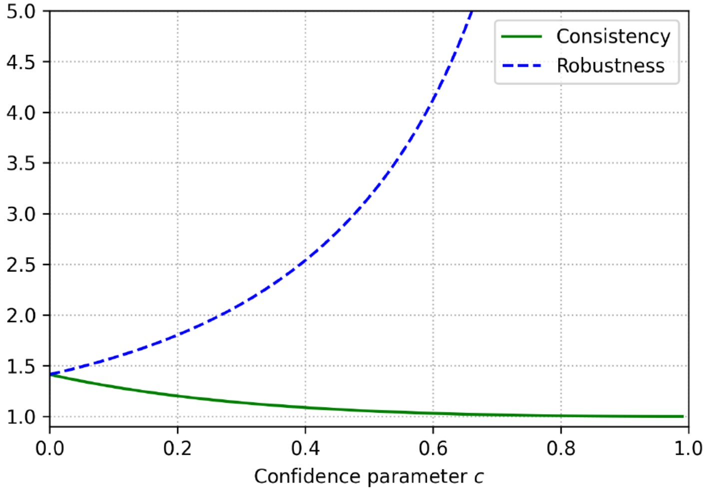
Coordinatewise Median with Predictions (CMP) mechanism

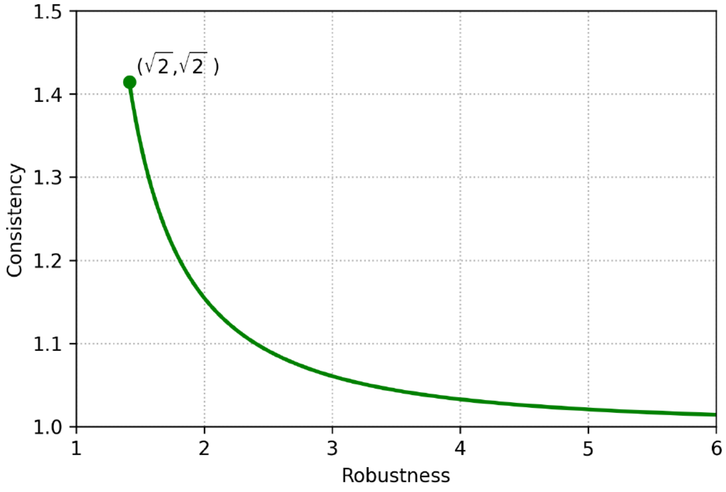
- c : confidence value in $[0,1)$
- P' : a multiset containing cn copies
- output $f(P, \hat{o}, c)$: $CM(P \cup P')$
- $\frac{\sqrt{2c^2+2}}{1+c}$ - consistent & $\frac{\sqrt{2c^2+2}}{1-c}$ - robust

Minimizing the utilitarian social cost

Coordinatewise Median with Predictions (CMP) mechanism

- $\frac{\sqrt{2c^2+2}}{1+c}$ - consistent & $\frac{\sqrt{2c^2+2}}{1-c}$ - robust

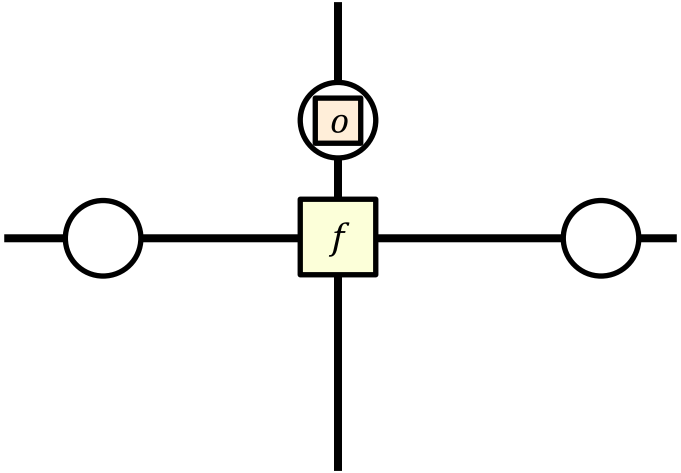




Minimizing the utilitarian social cost

$\mathcal{P}_{coa}(c)$: the class of all instances with prediction $\hat{o}$ and preferred points P such that

- output $f(P, \hat{o}, c) = (0,0)$
- optimal solution $o(P) = (0,1)$
- For $p \in P$, $p = (0,1)$ or $(x, 0)$ or $(-x, 0)$



$\mathcal{P}^{C}_{coa}(c)$: the subset of $\mathcal{P}_{coa}(c)$ where $\hat{o} = o(P)$

$\mathcal{P}^{R}_{coa}(c)$: the subset of $\mathcal{P}_{coa}(c)$ where $\hat{o} = (0,0)$

COA = **C**lusters-and-**O**pt-on-**A**xes

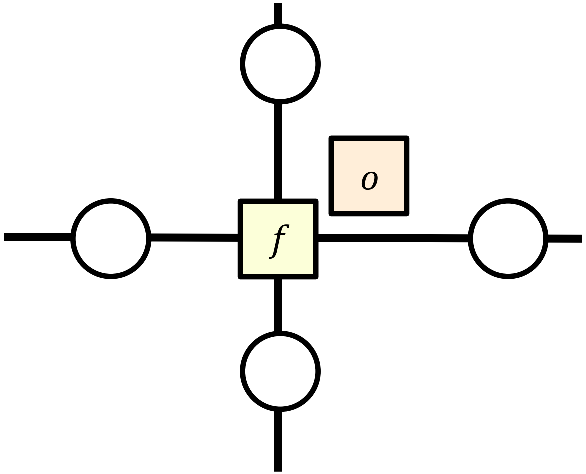
Minimizing the utilitarian social cost

Lemma : For any $c \in [0,1)$, the CMP mechanism with confidence c is α-consistent and β-robust, where $\alpha = \max_{(P,\hat{o}) \in \mathcal{P}^{C}_{coa}(c)} r(P, \hat{o} = o(P), c)$, and $\beta = \max_{(P,\hat{o}) \in \mathcal{P}^{R}_{coa}(c)} r(P, \hat{o} = (0,0), c)$.

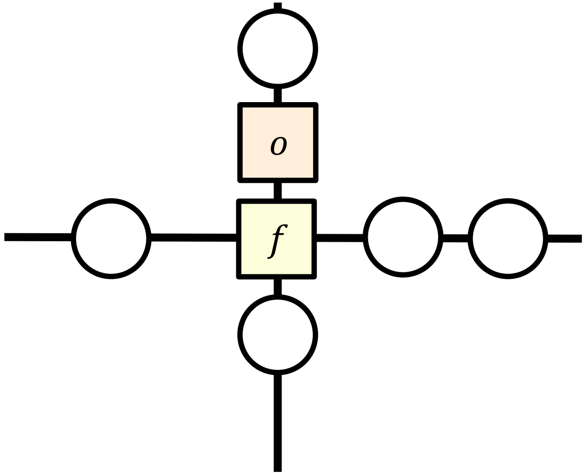
Minimizing the utilitarian social cost

CA instance : the points are all located at four clusters, one on each half-axis

OA instance : the points and the optimal location are all located on axis



CA



OA

Minimizing the utilitarian social cost

Lemma : For any $c \in [0,1)$, the CMP mechanism with confidence c is α-consistent and β-robust, where $\alpha = \max_{(P,\hat{o}) \in \mathcal{P}_{coa}^{C}(c)} r(P, \hat{o} = o(P), c)$, and $\beta = \max_{(P,\hat{o}) \in \mathcal{P}_{coa}^{R}(c)} r(P, \hat{o} = (0,0), c)$.

1. An arbitrary instance → either an instance in CA or OA
   (without improving the approximation ratio)
2. an instance in CA or OA → an instance in COA
   (without improving the approximation ratio)

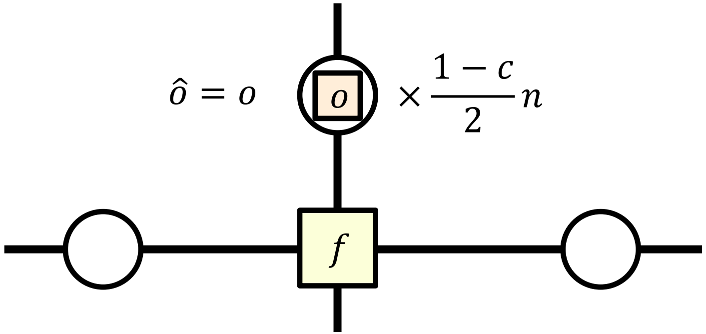
Minimizing the utilitarian social cost

Lemma : The CMP mechanism with parameter $c \in [0,1)$ is $\frac{\sqrt{2c^2+2}}{1+c}$-consistent and $\frac{\sqrt{2c^2+2}}{1-c}$-robust for the utilitarian objective.

Minimizing the utilitarian social cost

Lemma : The CMP mechanism with parameter $c \in [0,1)$ is $\frac{\sqrt{2c^2+2}}{1+c}$-consistent and $\frac{\sqrt{2c^2+2}}{1-c}$-robust for the utilitarian objective.

consistency maximize → the number of agents on $(0,1)$ is maximized : $\frac{1-c}{2}n$



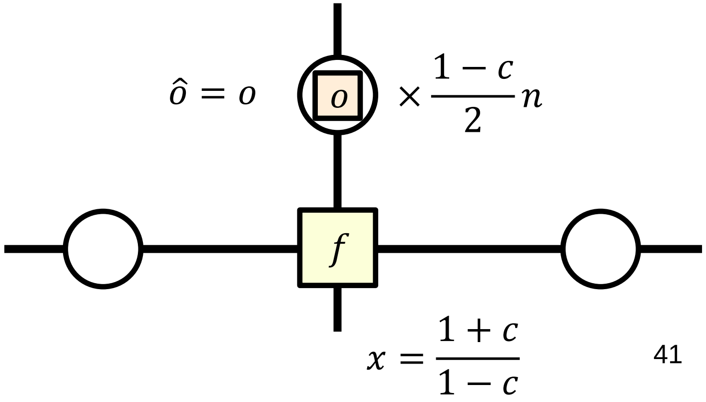
Minimizing the utilitarian social cost

Lemma : The CMP mechanism with parameter $c \in [0,1)$ is $\frac{\sqrt{2c^2+2}}{1+c}$-consistent and $\frac{\sqrt{2c^2+2}}{1-c}$-robust for the utilitarian objective.

consistency maximize → the number of agents on $(0,1)$ is maximized : $\frac{1-c}{2}n$

$$\frac{C^u(f,P)}{C^u(o,P)} = \frac{\frac{1+c}{2}n\cdot x + \frac{1-c}{2}n}{\frac{1+c}{2}n\cdot\sqrt{1+x^2}} = \frac{1-c+(1+c)x}{(1+c)\sqrt{1+x^2}} \rightarrow \text{maximum on } x = \frac{1+c}{1-c}$$

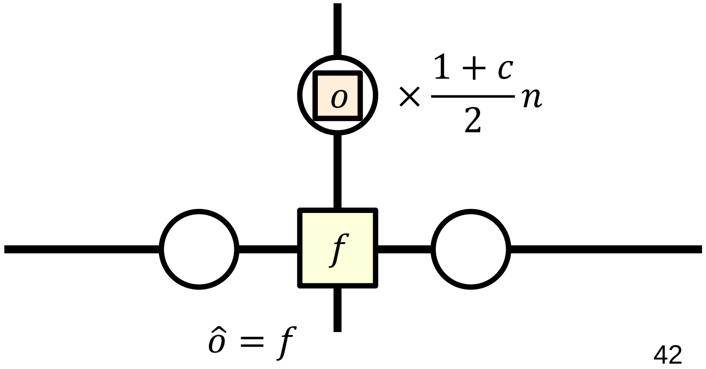
→ $\frac{\sqrt{2c^2+2}}{1+c}$-consistent



Minimizing the utilitarian social cost

Lemma : The CMP mechanism with parameter $c \in [0,1)$ is $\frac{\sqrt{2c^2+2}}{1+c}$-consistent and $\frac{\sqrt{2c^2+2}}{1-c}$-robust for the utilitarian objective.

robustness maximize → the number of agents on $(0,1)$ is maximized : $\frac{1+c}{2}n$



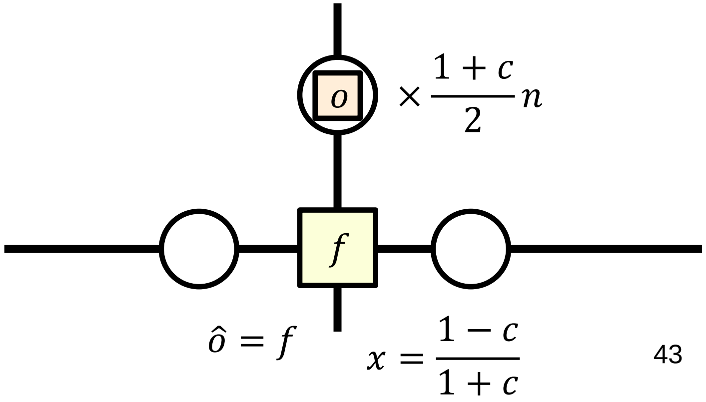
Minimizing the utilitarian social cost

Lemma : The CMP mechanism with parameter $c \in [0,1)$ is $\frac{\sqrt{2c^2+2}}{1+c}$-consistent and $\frac{\sqrt{2c^2+2}}{1-c}$-robust for the utilitarian objective.

robustness maximize → the number of agents on $(0,1)$ is maximized : $\frac{1+c}{2}n$

$$\frac{C^u(f,P)}{C^u(o,P)} = \frac{\frac{1-c}{2}n\cdot x+\frac{1+c}{2}n}{\frac{1-c}{2}n\cdot\sqrt{1+x^2}} = \frac{1+c+(1-c)x}{(1-c)\sqrt{1+x^2}}$$ → maximum on $x = \frac{1-c}{1+c}$

→ $\frac{\sqrt{2c^2+2}}{1-c}$ - robust



Minimizing the utilitarian social cost

Lemma : For CMP with confidence $c \in [0,1)$, there exists $(P, \hat{o}) \in \mathcal{P}^{C}_{coa}(c)$ such that

$r(P, \hat{o} = o(P), c) = \frac{\sqrt{2c^2+2}}{1+c}$, and $(Q, \hat{o}) \in \mathcal{P}^{R}_{coa}(c)$ such that $r(Q, \hat{o} = (0,0), c) = \frac{\sqrt{2c^2+2}}{1-c}$.

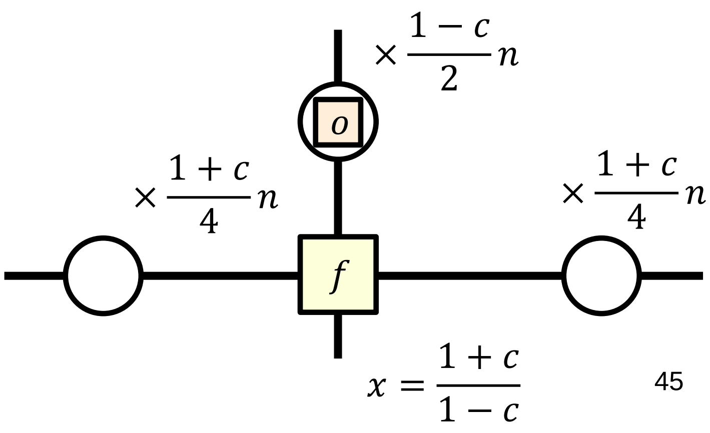
Minimizing the utilitarian social cost

Lemma : For CMP with confidence $c \in [0,1)$, there exists $(P, \hat{o}) \in \mathcal{P}^{C}_{coa}(c)$ such that

$r(P, \hat{o} = o(P), c) = \frac{\sqrt{2c^2+2}}{1+c}$, and $(Q, \hat{o}) \in \mathcal{P}^{R}_{coa}(c)$ such that $r(Q, \hat{o} = (0,0), c) = \frac{\sqrt{2c^2+2}}{1-c}$.

$C^u(o(P), P) = \frac{1+c}{2}\sqrt{1 + \left(\frac{1+c}{1-c}\right)^2}$, $C^u(f, P) = \frac{1+c}{2} \cdot \frac{1+c}{1-c} + \frac{1-c}{2}$

$\rightarrow r(P, \hat{o} = o(P), c) = \frac{\sqrt{2c^2+2}}{1+c}$



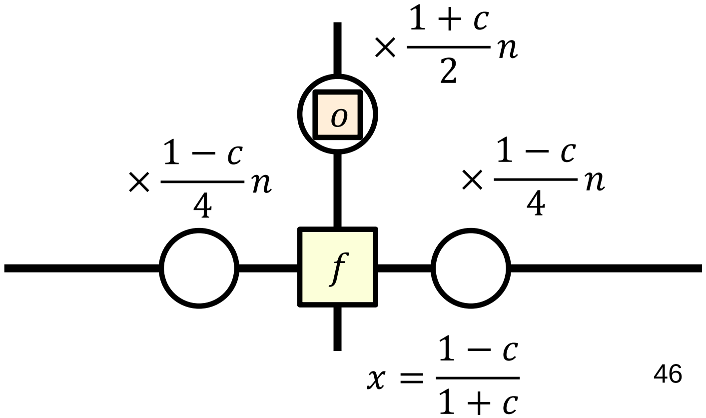
Minimizing the utilitarian social cost

Lemma : For CMP with confidence $c \in [0,1)$, there exists $(P, \hat{o}) \in \mathcal{P}^{C}_{coa}(c)$ such that

$r(P, \hat{o} = o(P), c) = \frac{\sqrt{2c^2+2}}{1+c}$, and $(Q, \hat{o}) \in \mathcal{P}^{R}_{coa}(c)$ such that $r(Q, \hat{o} = (0,0), c) = \frac{\sqrt{2c^2+2}}{1-c}$.

$C^u(o(Q), Q) = \frac{1-c}{2}\sqrt{1 + \left(\frac{1-c}{1+c}\right)^2}$, $C^u(f, Q) = \frac{1-c}{2} \cdot \frac{1-c}{1+c} + \frac{1+c}{2}$

$\rightarrow r(Q, \hat{o} = (0,0), c) = \frac{\sqrt{2c^2+2}}{1-c}$



Minimizing the utilitarian social cost

Theorem : For any deterministic and strategyproof mechanism that guarantees a consistency of $\frac{\sqrt{2c^2+2}}{1+c}$, for some constant $c \in (0,1)$, its robustness is no better than $\frac{\sqrt{2c^2+2}}{1-c}$ for the utilitarian objective.

Minimizing the utilitarian social cost

Theorem：The CMP mechanism with parameter $c \in [0,1)$ achieves a $\min\left\{\frac{\sqrt{2c^2+2}}{1+c} + \eta, \frac{\sqrt{2c^2+2}}{1-c}\right\}$ - approximation, where η is the prediction error, for the utilitarian objective.