Fixed-price approximations in bilateral trade

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Bilateral Trade

Bilateral Trade

- Two agents: a buyer and a seller
- buyer's valuation $B \sim F_B$, seller's valuation $S \sim F_S$

Bilateral Trade Mechanisms

- Allocation function $A : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$. $A(r, s) = 1$ if a transaction should occur, 0 otherwise.
- Payment function $\Pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ If a trade occurs, it determines the price.

Bilateral Trade

We measure the performance of a mechanism by two values:

1. Gains from trade(GFT): $B - S$ wherever trade occurs and 0 otherwise.
2. welfare: B if trade occurs, S otherwise.

Note that (welfare) - (GFT) $= S$ always holds. The "first-best" optimum is considered to be $\max(B, S)$ for welfare, and $\max(B - S, 0)$ for GFT.

Prior work

Theorem(Myerson and Satterthwaite)

No individually rational Bayesian incentive-compatible (BIC) mechanism attains the first-best optimum in general.

Prior work

We'll use notation used in paper of Myerson and Satterthwaite for describing their results.

Bilateral Trading Problem

- individual 1 owns an object, individual 2 wants to buy.
- Each individual's valuation is V_1, V_2 $(= S, B)$.
- V_i is distributed over a given interval $[a_i, b_i]$
- density function f_i for V_i is continuous and positive on $[a_i, b_i]$
- F_i be cumulative distribution functions corresponding to f_i $(F_1 = F_S, F_2 = F_B)$
- each individual knows her own valuation, but she considers the other's valuation as a random variable. That is, individual 1 knows V_1 and F_2.

Prior work

Direct bargaining mechanism

- Each individual simultaneously reports her valuation.
- Direct mechanism is characterized by two outcome functions p, x.
- $p(v_1, v_2)$: the probability that the trade occurs when reported valuation are v_1, v_2 $(= A(s, b))$
- $x(v_1, v_2)$: the expected payment from buyer to seller when reported valuation are v_1, v_2 $(= \Pi(s, b))$

A direct mechanism is Bayesian incentive-compatible(BIC) if honest reporting forms a Bayesian Nash equilibrium. That is, in an incentive-compatible mechanism, each individual can maximize his expected utility by reporting his true valuation, given that the other is expected to report honestly.

Prior work

Revelation principle

For any Bayesian equilibrium of any bargaining game, there is an equivalent **incentive-compatible direct mechanism** that always yields the same outcome.

Therefore, without any loss of generality, we can restrict our attention to incentive-compatible direct mechanisms.

Prior work

We'll consider direct mechanisms satisfies following conditions:

Requirements

- Individual Rationality(IR): each individual's expected gain should be nonnegative.
- Bayesian Incentive Compatibility(BIC)

Prior work

Define following quantities:

Definition

- $\bar{x_1}(v_1) = \int_{a_2}^{b_2} x(v_1, t_2) f_2(t_2) dt_2,\ \bar{x_2}(v_2) = \int_{a_1}^{b_1} x(t_1, v_2) f_1(t_1) dt_1$
- $\bar{p_1}(v_1) = \int_{a_2}^{b_2} p(v_1, t_2) f_2(t_2) dt_2,\ \bar{p_2}(v_2) = \int_{a_1}^{b_1} p(t_1, v_2) f_1(t_1) dt_1$
- $U_1(v_1) = \bar{x_1}(v_1) - v_1 \bar{p_1}(v_1),\ U_2(v_2) = v_2 \bar{p_2}(v_2) - \bar{x_2}(v_2)$

$U_1(v_1)$: expected gains from trade for seller if her valuation is v_1
$U_2(v_2)$: expected gains from trade for buyer if her valuation is v_2.

IR: $U_1(v_1) \geq 0, U_2(v_2) \geq 0$ for all v_1, v_2
BIC: $U_1(v_1) \geq \bar{x_1}(\hat{v_1}) - v_1 \bar{p_1}(\hat{v_1})$, $U_2(v_2) \geq v_2 \bar{p_2}(\hat{v_2}) - \bar{x_2}(\hat{v_2})$ for true valuation v_i and arbitrary $\hat{v}_i$.

Prior work

Lemma

For any BIC mechanism, $U_1(b_1) = \min_{v_1} U_1(v_1), U_2(a_2) = \min_{v_2} U_2(v_2)$

Proof. For every two possible valuation $v_1, \hat{v_1}$ for seller,
$U_1(v_1) = \bar{x_1}(v_1) - v_1\bar{p_1}(v_1) \geq \bar{x_1}(\hat{v_1}) - v_1\bar{p_1}(\hat{v_1})$
$U_1(\hat{v_1}) = \bar{x_1}(\hat{v_1}) - \hat{v_1}\bar{p_1}(\hat{v_1}) \geq \bar{x_1}(v_1) - \hat{v_1}\bar{p_1}(v_1)$
Therefore, $(\hat{v_1} - v_1)\bar{p_1}(v_1) \geq U_1(v_1) - U_1(\hat{v_1}) \geq (\hat{v_1} - v_1)\bar{p_1}(\hat{v_1})$.

From this, $\bar{p_1}$ is decreasing and $U_1'(v_1) = -\bar{p_1}(v_1)$ that
$U_1(v_1) = U_1(b_1) + \int_{v_1}^{b_1} \bar{p_1}(t_1)dt_1$ is decreasing. Similarly,
$U_2(v_1) = U_2(a_2) + \int_{a_2}^{v_2} \bar{p_2}(t_2)dt_2$ is increasing.

Therefore, $U_1(b_1) = \min_{v_1} U_1(v_1)$ and $U_2(a_2) = \min_{v_2} U_2(v_2)$.

Prior work

Following lemma holds:

Lemma

For any BIC mechanism,

$$U_1(b_1) + U_2(a_2) = \min_{v_1} U_1(v_1) + \min_{v_2} U_2(v_2) =$$

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} [f_1(v_1)(v_2 f_2(v_2) - (1 - F_2(v_2)) - f_2(v_2)(v_1 f_1(v_1) - F_1(v_1))] \cdot p(v_1, v_2) dv_1 dv_2.$$

It needs some calculation to derive above lemma. This lemma shows that $U_1(b_1) + U_2(a_2)$ depends only on F_i and p_i, not x_i.

Prior work

Ex post efficiency

A mechanism (p, x) is ex post efficient if and only if
$p(v_1, v_2) = 1$ if $v_1 < v_2$,
$p(v_1, v_2) = 0$ if $v_2 > v_1$

An ex post efficient mechanism attains the first-best optimum, and $\bar{p_1}(v_1) = 1 - F_2(v_1)$, $\bar{p_2}(v_2) = F_1(v_2)$ holds.

Prior work

Assume that $\max(a_1, a_2) < \min(b_1, b_2)$. That is, two individuals' valuation domain intersects. For an ex-post efficient mechanism,

$$U_1(b_1) + U_2(a_2) =$$

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} [f_1(v_1)(v_2 f_2(v_2) - (1 - F_2(v_2)) - f_2(v_2)(v_1 f_1(v_1) - F_1(v_1))] \cdot p(v_1, v_2) dv_1 dv_2$$

$$= -\int_{a_2}^{b_1} (1 - F_2(t))F_1(t)dt < 0$$

So it cannot be individually rational, and the theorem is now proved:

Myerson-Satterthwaite theorem

if (a_1, b_1) and (a_2, b_2) intersects, then no Bayesian incentive-compatible individually rational mechanism can be ex post efficient.

Bilateral Trade

Now, return to the original paper.

We are interested in only incentive-compatible mechanisms (it is enough because of the revelation principle)

Incentive Compatiblity

- Bayesian incentive compatibility(BIC): Reporting true values should be an optimal strategy for each agent, **in expectation**.
- Dominant-strategy incentive compatibility(DSIC): Reporting true values is **always** an optimal strategy for all agents.

So, DSIC mechanisms are Bayesian incentive compatible, but converse does not hold.

Bilateral Trade

For this paper, our consideration will be exclusively on mechanisms that satisfies DSIC and IR.

Since DSIC mechanisms are BIC mechanisms, so there is no IR DSIC mechanism attains first-best optimum.

Our goal: make DSIC mechanism that welfare and gain for trade close to optimum as possible.

Results

Variant	Welfare approximation		Gains from trade approximation	
Symmetric, full knowledge	$(2+\sqrt{2})/4$*		1/2*	[11, 12]
Symmetric, 1 prior sample	3/4*		1/2*	
Asymmetric, full knowledge	$1-1/e+\varepsilon$		0*	[3]
Asymmetric, 1 prior sample	1/2	[8]	0*	[3]

Fixed-price Mechanisms

By definition, DSIC mechanisms satisfy the following:

- $\Pi(s,b) - s \cdot A(s,b) \geq \Pi(s',b) - s \cdot A(s',b)$
- $b \cdot A(s,b) - \Pi(s,b) \geq b \cdot A(s,b') - \Pi(s,b')$

Theorem

DSIC mechanisms for bilateral trade are essentially **fixed-price mechanisms**, *where the Π is a single value p that is only related to F_S and F_B, not valuations. And $A(b,s) = \mathbf{1}_{s \leq p \leq b}$, trade occurs if and only if $s \leq p \leq b$.*

GFT and Welfare for symmetric case

Consider only symmetric case: $F = F_B = F_S$.

For a fixed-price mechanism with price p,

- Optimal gains from trade OPT-$GFT(F) = \mathbb{E}[\mathbf{1}_{B>S}(B - S)]$
- Gains from trade $GFT(p, F) = \mathbb{E}[\mathbf{1}_{B \geq p > S}(B - S)]$
- Optimal welfare
  OPT-$W(F) = \mathbb{E}[S] + \mathbb{E}[\mathbf{1}_{B>S}(B - S)] = \mathbb{E}[S] +$ OPT-$GFT(F)$
- welfare $W(p, F) = \mathbb{E}[S] + GFT(p, F)$

GFT value for symmetric case

Let f be density function of F. Then following holds:

$$
\begin{aligned}
\text{OPT-}GFT(F) &= \mathbb{E}[\mathbf{1}_{B>S}(B-S)] \\
&= \int_0^\infty \mathbf{1}_{S\le x<B} dx \\
&= \int_0^\infty F(x)(1-F(x))dx
\end{aligned}
$$

$$
\begin{aligned}
GFT(p,F) &= \mathbb{E}[\mathbf{1}_{B\ge p>S}(B-S)] \\
&= \mathbb{E}[\mathbf{1}_{B\ge p>S}(B-p)] + \mathbb{E}[\mathbf{1}_{B\ge p>S}(p-S)] \\
&= \mathbb{E}[\mathbf{1}_{B\ge p}(B-p)]Pr(S<p) + \mathbb{E}[\mathbf{1}_{p>S}(p-S)]Pr(B\ge p) \\
&= F(p)\int_p^\infty (1-F(x))dx + (1-F(p))\int_0^p F(x)dx
\end{aligned}
$$

Single Sample Approximation

Theorem

The symmetric bilateral trade mechanism which under a valuation distribution F posts a price $p \sim F$ achieves exactly $1/2$ of the OPT-GFT.

Proof. $\mathbb{E}_{p\sim F}[GFT(p,F)]$
$= \int_0^\infty \left[F(p)\int_p^\infty (1-F(x))dx + (1-F(p))\int_0^p F(x)dx\right] f(p)dp$
Let $\gamma_1 = \int_0^\infty f(p)F(p)\int_p^\infty (1-F(x))dxdp$,
$\gamma_2 = \int_0^\infty f(p)(1-F(p))\int_0^p F(x)dxdp$,
then $\mathbb{E}_{p\sim F}[GFT(p,F)] = \gamma_1 + \gamma_2$ holds.

$$\therefore \mathbb{E}_{p\sim F}[GFT(p,F)] = \gamma_1 + \gamma_2 = \frac{1}{2}\int_0^\infty F(x)(1-F(x))dx$$

which is OPT-GFT$(F)/2$.

Single Sample Approximation

Theorem

The symmetric bilateral trade mechanism which under a valuation distribution F posts a price $p \sim F$ achieves exactly $1/2$ of the OPT-GFT.

Proof.

$$\gamma_1 = \int_0^\infty f(p)F(p)\int_p^\infty (1-F(x))dxdp$$

$$= \frac{1}{2}\int_0^\infty F(p)^2(1-F(p))dp \text{ (some calculations are omitted)}$$

$$\gamma_2 = \int_0^\infty f(p)(1-F(p))\int_0^p F(x)dxdp$$

$$= \frac{1}{2}\int_0^\infty F(p)(1-F(p))^2 dp$$

$$\therefore \mathbb{E}_{p\sim F}[GFT(p,F)] = \gamma_1 + \gamma_2 = \frac{1}{2}\int_0^\infty F(x)(1-F(x))dx$$

which is OPT-GFT$(F)/2$.

Single Sample Approximation

Theorem

The symmetric bilateral trade mechanism which under a valuation distribution F posts a price $p \sim F$ achieves a $3/4$-approximation of the optimal welfare.

Proof.

$$\frac{\mathbb{E}_{p\sim F}[GFT(p,F)]}{\text{OPT-W}(F)} = \frac{\mu + \mathbb{E}_{p\sim F}[GFT(p,F)]}{\mu + \text{OPT-GFT}(F)} = \frac{\mu + \text{OPT-GFT}(F)/2}{\mu + \text{OPT-GFT}(F)}$$

On the other hand,

$$OPT - GFT(F) = \int_0^\infty F(x)(1-F(x))dx \leq \int_0^\infty 1\cdot(1-F(x))dx$$

$= \int_0^\infty Pr[t \geq x]_{t\sim F} = \mu.$

$$\therefore \frac{\mathbb{E}_{p\sim F}[GFT(p,F)]}{\text{OPT-W}(F)} \geq \frac{3}{4}$$

.

Best-Possible Approximation of Welfare

Assume F is known, find best possible p for maximizing welfare.

Theorem

$p = \mu$ is optimal. That is, $p^* = \mathbb{E}[S] = \mathbb{E}[B]$

Proof.

$$\begin{aligned} W(p, F) &= \mathbb{E}[S] + \mathbb{E}[\mathbf{1}_{B \geq p > S}(B - S)] \\ &= \mathbb{E}[S] + \mathbb{E}[B \cdot \mathbf{1}_{B > p}] \cdot F(p) - \mathbb{E}[S \cdot \mathbf{1}_{S \leq p}](1 - F(p)) \\ &= \mathbb{E}[S] + (\mathbb{E}[S] - \mathbb{E}[S \cdot \mathbf{1}_{S \leq p}]) \cdot F(p) - \mathbb{E}[S \cdot \mathbf{1}_{S \leq p}](1 - F(p)) \\ &= \mathbb{E}[S](1 + F(p)) - \mathbb{E}[S \cdot \mathbf{1}_{S \leq p}] \\ &= \mathbb{E}[S](1 + F(p)) - pF(p) + \int_0^p F(s)ds \end{aligned}$$

Best-Possible Approximation of Welfare

Assume F is known, find best possible p for maximizing welfare.

Theorem

$p = \mu$ is optimal. That is, $p^* = \mathbb{E}[S] = \mathbb{E}[B]$

Proof.

$$W(p, F) = \mathbb{E}[S](1 + F(p)) - pF(p) + \int_0^p F(s)ds$$

$$\frac{dW}{dp} = \mathbb{E}[S]f(p) - F(p) - pf(p) + F(p) = (\mathbb{E}[S] - p)f(p)$$

Therefore, $W(p, F)$ is maximized when $\mu = \mathbb{E}[S] = p$.

Best-Possible Approximation of Welfare

We have seen $W(p, F)$ is maximized when $p = \mu$. Therefore, best-possible approximation ratio is

$$\inf_F \frac{W(\mu_F, F)}{\text{OPT-}W(F)}$$

On the other hand,

$$\begin{aligned} W(\mu, F) &= \mu \cdot (1 + F(\mu)) - \mathbb{E}[S \cdot \mathbf{1}_{S \leq \mu}] \\ &= \mu + (\mu - \mathbb{E}[S|S \leq \mu]) \cdot F(\mu) \end{aligned}$$

$W(\mu, F)$ depends on only three quantities: μ, $F(\mu)$, $\mathbb{E}[S|S \leq \mu]$.

Best-Possible Approximation of Welfare

$W(\mu, F)$ depends on only three quantities: μ, $F(\mu)$, $\mathbb{E}[S|S \leq \mu]$.

Define subspace of probability distributions that fixes the three quantities:

$\Delta(\mu, \mu_1, \gamma) :=$

$$\{\text{probability distribution } F \mid \mathbb{E}[S] = \mu, \mathbb{E}[S|S \leq \mu] = \mu_1, F(\mu) = \gamma\}$$

Then, the approximation ratio is equal to

$$\inf_{0 \leq \mu_1 \leq \mu; \mu > 0; 0 < \gamma \leq 1} \left[\inf_{F \in \Delta(\mu, \mu_1, \gamma)} \frac{\mu + (\mu - \mu_1)\gamma}{\text{OPT-}W(F)} \right]$$

.

Best-Possible Approximation of Welfare

Let $\Delta^4(\mu, \mu_1, \gamma)$ be a subset of $\Delta(\mu, \mu_1, \gamma)$ which is a set of distributions supported on **at most 4 points**. Then the following lemma holds:

Lemma

For any fixed $0 \leq \mu_1 \leq \mu \leq 1, \mu > 0, 0 < \gamma \leq 1$,

$$\inf_{F \in \Delta(\mu,\mu_1,\gamma)} \frac{\mu + (\mu - \mu_1)\gamma}{\textit{OPT-W}(F)} = \inf_{F \in \Delta^4(\mu,\mu_1,\gamma)} \frac{\mu + (\mu - \mu_1)\gamma}{\textit{OPT-W}(F)}$$

Therefore, to get best possible approximation ratio, it is enough to consider only distributions of at most 4 points.

Best-Possible Approximation of Welfare

Sketch of proof.

First, Rescaling the domain of F in order to $F(0) = 0, F(1) = 1$.

- For any probability mass in $(0, \mu)$, split the mass into two equal masses. Move each mass in opposite directions, until one mass hits the boundary of the interval $[0, \mu]$.
- For any probability mass in $(\mu, 1)$, split the mass into two equal masses. Move each mass in opposite directions, until one mass hits the boundary of the interval $[\mu + \delta, 1]$ (sufficiently small $\delta > 0$).

Above operation do not change μ, $F(\mu)$, $\mathbb{E}[S|S \leq \mu]$ and increase $\mathbb{E}[B - S]$ and therefore OPT-$W(F)$. So there is always *better* distribution consists of 4 points.

Best-Possible Approximation of Welfare

By using former lemma and large amount of calculation, we can derive the following result:

Theorem

$$\inf_{0\le\mu_1\le\mu;\mu>0;0<\gamma\le1}\left[\inf_{F\in\Delta^4(\mu,\mu_1,\gamma)}\frac{\mu+(\mu-\mu_1)\gamma}{\mathit{OPT}\text{-}W(F)}\right]\ge\frac{2+\sqrt{2}}{4}$$

And also, there is a sequence of distributions $\{F_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty}\frac{W(\mu_n,F_n)}{\text{OPT-}W(F_n)}=\frac{2+\sqrt{2}}{4}$$

Therefore, the following holds:

Theorem

the best-possible approximation ratio of welfare is $\frac{2+\sqrt{2}}{4}$.

Asymmetric Case

- Given F_B and F_S, known best approximation ratio for welfare was $1 - 1/e$.
- It is based on the fact

$$\sup_p \frac{W(p, F_S, F_B}{\text{OPT-W}(F_S, F_B)} \geq 1 - \frac{1}{e} + \frac{1}{e} \cdot \mathbb{E}[\max(S - B, 0)]$$

- It is also known that $\frac{3}{4}$-approximation is possible for distribution (F_S, F_B) that satisfies $\mathbb{E}[\max(S - B, 0)] = 0$.
- Using above two facts, $1 - 1/e + \epsilon$ bound can be achieved heuristically using the closeness of $\mathbb{E}[\max(S - B, 0)]$ to 0 $(\epsilon \geq 0.0001)$.

Asymmetric Case

- Yang Cai's paper relesed in 2023 shows there is a fixed-price mechanism achives at least 0.72 of the optimal welfare, and there is a not fixed-price mechanism achieves 0.7381 of the optimal welfare.
- Just like we used distribution with only 4 possible values in approximation of welfare in symmetric case, it uses discretization technique.

Thank you!