# The Rate Distortion Theory

Yonsei CS Theory Student Group Seminar

Information Theory Series, Week 5

Presented by Sungmin Kim on 24' Apr. 04

# Motivation

High loss

Low loss

Topics to consider

- Definition of loss (=distortion)
- Relation between rate and loss
- Methods for computing optimal rate given loss

# Rate and Distortion

**Situation**

$X^n$ → encoder $\xrightarrow{f_n(x^n) \in \{1, \dots, 2^{nR}\}}$ decoder → $\hat{X}^n$

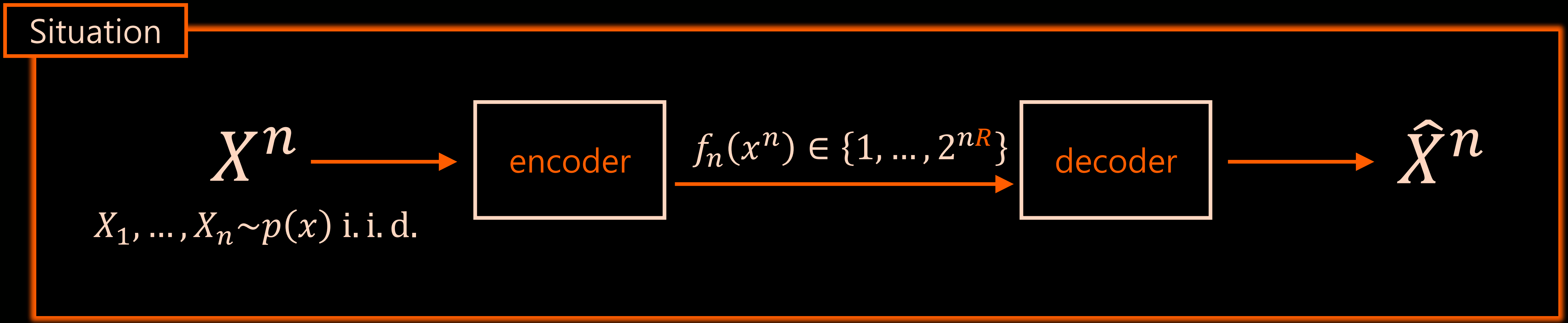
$X_1, \dots, X_n \sim p(x)$ i.i.d.

- [Rate] Define rate $R$ as the average number of bits per symbol for representing $X^n$.
  - ▷ Intuitively, we lose information as $R$ decreases

Encode 4 pixels using a single pixel

$R = 1/4$

# Rate and Distortion




- [ ] [Rate] Recall that, in the channel coding theorem, the definition of the rate $R$ is

$$R = \frac{\log M}{n},$$

where $M$ is the number of possible values of the channel input. Rearranging gives

$$M = 2^{nR}.$$

# Rate and Distortion

**Situation (lossy encoding)**

$X^n$ → encoder → $f_n(x^n) \in \{1, \ldots, 2^{nR}\}$ → decoder → $\hat{X}^n$

$R$ is a parameter

$X_1, \ldots, X_n \sim p(x)$ i.i.d.

**Situation (Channel Coding Thm)**

encoder → $X^n : \{1, \ldots, M\} \to \mathcal{X}$ → Discrete Memoryless Channel → $Y^n : \mathcal{Y} \to \{1, \ldots, M\}$ → decoder

Fixed!

# Rate and Distortion

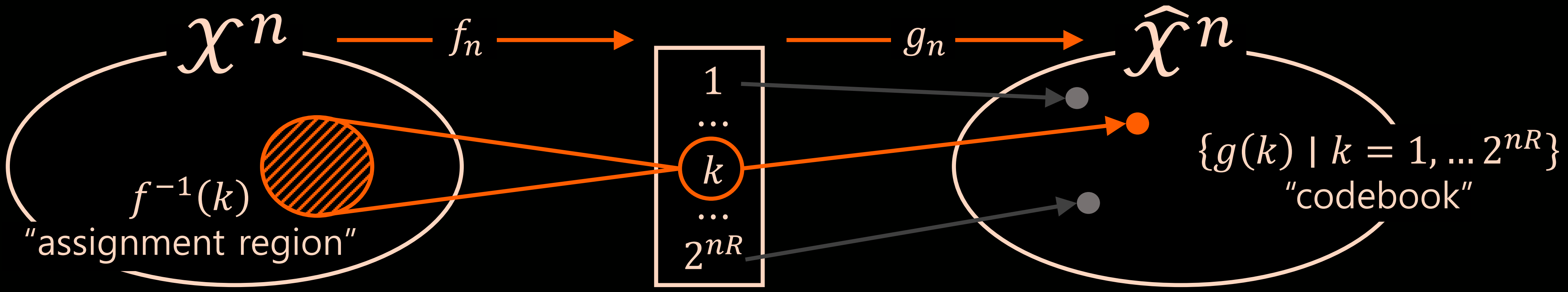
**Situation**

$$X^n \longrightarrow \boxed{\text{encoder}} \xrightarrow{f_n(x^n) \in \{1, \dots, 2^{nR}\}} \boxed{\text{decoder}} \longrightarrow \hat{X}^n$$

$X_1, \dots, X_n \sim p(x)$ i.i.d.

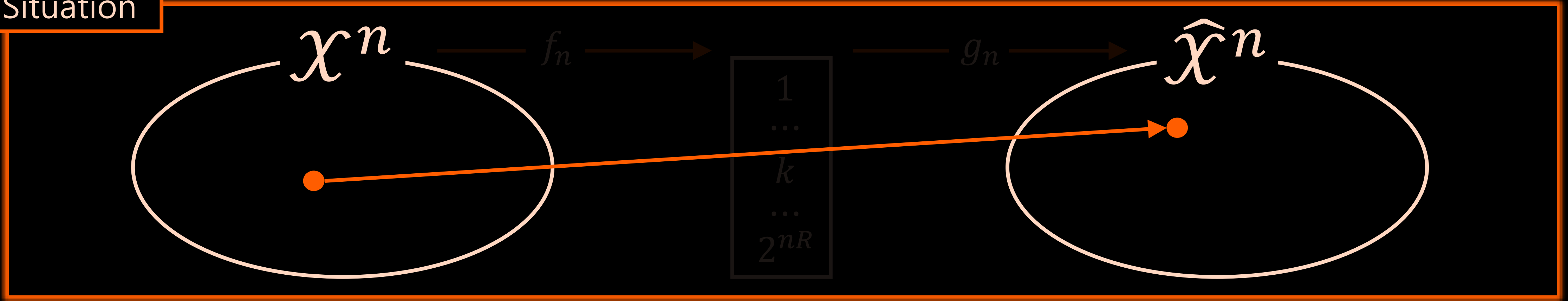
- [Distortion Code] Given a rate $R$, a function pair $(f_n, g_n)$ where
  - the encoding function $f_n : \mathcal{X}^n \to \{1,2,\dots,2^{nR}\}$
  - the decoding function $g_n : \{1,2,\dots,2^{nR}\} \to \hat{\mathcal{X}}^n$

  is a $(2^{nR}, n)$-rate distortion code, i.e., the lossy encoding scheme.



# Rate and Distortion

Situation




- [Distortion] Define distortion function $d: \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}^+$:
  - represents how different $\hat{X}$ is from $X$
  - options include Hamming distortion

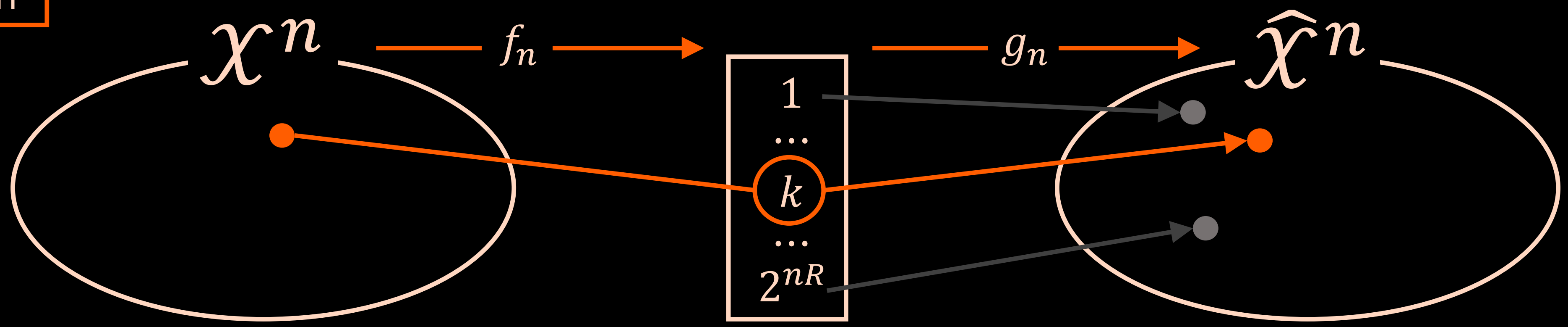
$$[d(x, \hat{x}) = 0] \Leftrightarrow [x = \hat{x}]$$

  - Or the squared-error distortion

$$d(x, \hat{x}) = (x - \hat{x})^2.$$

# Rate and Distortion

Situation




- [Distortion between sequences] The distortion between $x^n$ and $\hat{x}^n$ is

$$d(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, \hat{x}_i).$$

- [Distortion of a code] The distortion $D$ of a $(2^{nR}, n)$-rate distortion code $(f_n, g_n)$ is

$$D = \mathbb{E}\left[d\left(X^n, g_n\big(f_n(X^n)\big)\right)\right],$$

the expected distortion over all $X^n$ values.

# Rate and Distortion

**Situation**

$$X^n \longrightarrow \boxed{\text{encoder}} \xrightarrow{f_n(x^n) \in \{1, \dots, 2^{nR}\}} \boxed{\text{decoder}} \longrightarrow \hat{X}^n$$

$X_1, \dots, X_n \sim p(x)$ i. i. d.

- [Achievability] The rate-distortion pair $(R, D)$ is achievable if there exists a sequence of $(2^{nR}, n)$-rate distortion codes $(f_n, g_n)$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[d\left(X^n, g_n\left(f_n(X^n)\right)\right)\right] \leq D.$$

- [Rate Distortion Function] The rate distortion function $R(D)$ gives the infimum of rates $R$ such that $(R, D)$ is in the closure of the set of achievable rate distortion pairs.

# The Information Rate Distortion Function

Def The information rate distortion function $R^{(I)}(D)$ is defined as the following:

$$\min_{p(x,\hat{x}):\sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x})\leq D} I(X;\hat{X})$$

i.e., the minimum mutual information over all joint distributions $p(x,\hat{x})$ with total distortion at most $D$.

Thm 10.2.1 The minimum achievable rate at distortion $D$ is exactly

$$R(D) = R^{(I)}(D).$$

- [part 1] $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.
- [part 2] $\left(R^{(I)}(D), D\right)$ is achievable.

# The Minimum Achievable Rate for Distortion

Prove $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

Lem $R^{(I)}(D)$ is convex and non-increasing in $D$.

- [non-increasing] if $D$ increases, more joint distributions $p(x,\hat{x})$ should be considered;

$$\min_{p(x,\hat{x}):\sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x})\leq D} I(X;\hat{X})$$

Thus, $R^{(I)}(D)$ is non-increasing in $D$.

# The Minimum Achievable Rate for Distortion

Prove $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

Lem $R^{(I)}(D)$ is convex and non-increasing in $D$.

[convexity] First, rewrite

$$D = \mathbb{E}\left[d\left(X^n, g_n\left(f_n(X^n)\right)\right)\right] = \sum_{(x,\hat{x})} p(x^n, \hat{x}^n) d(x^n, \hat{x}^n)$$

i.e., $D$ is linear in $p(\hat{x}^n \mid x^n)$.

Now, consider $(R_1, D_1)$ and $(R_2, D_2)$, both on the rate distortion curve.

Let $p_1(x, \hat{x}) = p(x)p_1(\hat{x} \mid x)$ and $p_2(x, \hat{x}) = p(x)p_2(\hat{x} \mid x)$ be their distributions, resp.

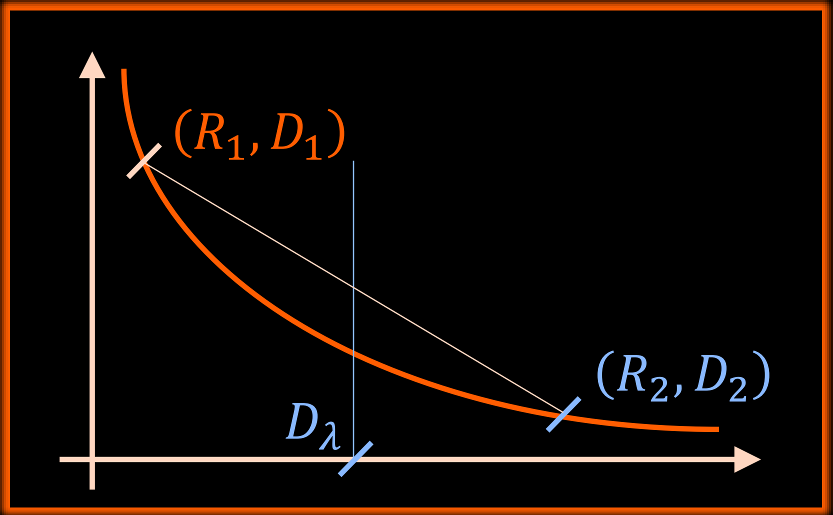
Let $p_\lambda = \lambda p_1 + (1-\lambda)p_2$ and we have $D_\lambda = \lambda D_1 + (1-\lambda)D_2$ by linearity in $p(\hat{x}^n \mid x^n)$.

# The Minimum Achievable Rate for Distortion

Prove $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

Lem $R^{(I)}(D)$ is convex and non-increasing in $D$.

☐ [convexity-cont.] Recall that $I(X;\hat{X})$ is convex (Thm. 2.7.4)




$$
\begin{aligned}
R^{(I)}(D_\lambda) &\leq I_{p_\lambda}(X;\hat{X}) \\
&\leq \lambda I_{p_1}(X;\hat{X}) + (1-\lambda) I_{p_2}(X;\hat{X}) \\
&= \lambda R(D_1) + (1-\lambda) R(D_2)
\end{aligned}
$$

Thus, $R^{(I)}(D)$ is convex in $D$.

# The Minimum Achievable Rate for Distortion

$\boxed{\text{Prove}}$ $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

- Follow the series of inequalities:

$$
\begin{aligned}
nR &\geq H\big(f_n(X^n)\big) && \text{[Property of } H\text{] } H(X) \leq \log|\mathcal{X}| \\
&\geq H\big(f_n(X^n)\big) - H\big(f_n(X^n) \mid X^n\big) \\
&= I\big(X^n; f_n(X^n)\big) && \text{[Definition of mutual information]} \\
&\geq I\big(X^n; \hat{X}^n\big) && \text{[Data processing inequality]} \\
&= H(X^n) - H\big(X^n \mid \hat{X}^n\big) \\
&= \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H\big(X_i \mid \hat{X}^n, X_{i-1}, \dots, X_1\big) && \text{[}X_i\text{'s independent] [Chain rule]}
\end{aligned}
$$

# The Minimum Achievable Rate for Distortion

$\boxed{\text{Prove}}$ $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

- Follow the series of inequalities:

$$nR \geq \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H\left(X_i \mid \hat{X}^n, X_{i-1}, \dots, X_1\right)$$

$$\geq \sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n} H\left(X_i \mid \hat{X}_i\right) \quad \cdots\cdots \text{[Conditioning reduces entropy]}$$

$$= \sum_{i=1}^{n} I\left(X_i; \hat{X}_i\right)$$

# The Minimum Achievable Rate for Distortion

Prove $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

- Follow the series of inequalities:

$$nR \geq \sum_{i=1}^{n} I\left(X_i; \hat{X}_i\right)$$

$$\geq \sum_{i=1}^{n} R^{(I)}\left(\mathbb{E}\left[d\left(X_i, \hat{X}_i\right)\right]\right) \quad \text{[Definition of } R^{(I)}(D)\text{]}$$

$$\geq nR^{(I)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}\left[d\left(X_i, \hat{X}_i\right)\right]\right)\right) \quad \text{[Convexity of } R^{(I)}(D)\text{]}$$

# The Minimum Achievable Rate for Distortion

Prove $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

- Follow the series of inequalities:

$$nR \geq nR^{(I)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{E}\left[d\left(X_i, \hat{X}_i\right)\right]\right)\right)$$

$$= nR^{(I)}\left(\mathbb{E}\left[d\left(X, \hat{X}\right)\right]\right) \quad \text{[Definition of } d(X, \hat{X})\text{]}$$

$$\geq nR^{(I)}(D) \quad [R^{(I)}(D) \text{ non-increasing}] \; [\mathbb{E}[d(X, \hat{X})] \leq D \text{ from condition}]$$

- Therefore, we have $R \geq R^{(I)}(D)$ for any $(2^{nR}, n)$-rate distortion code with distortion $\leq D$.

# The Minimum Achievable Rate for Distortion

**Prove** $\left(R^{(I)}(D), D\right)$ is achievable.

**Claim** For any $\delta > 0$, there exists a rate distortion code with rate $R$ and distortion $\leq D + \delta$.

**Proof** Technical; uses the distortion $\epsilon$-typicality to bound probabilities

- $\left|-\frac{1}{n}\log p(x^n) - H(X)\right| < \epsilon,$
- $\left|-\frac{1}{n}\log p(\hat{x}^n) - H(\hat{X})\right| < \epsilon,$
- $\left|-\frac{1}{n}\log p(x^n, \hat{x}^n) - H(X, \hat{X})\right| < \epsilon,$
- $\left|d(x^n, \hat{x}^n) - \mathbb{E}\left[d(X, \hat{X})\right]\right| < \epsilon.$

# Characterizing the Rate Distortion Function

**Prob** Compute the rate distortion function

$$R(D) = \min_{q(\hat{x}|x):\sum_{(x,\hat{x})} p(x)q(\hat{x}|x)d(x,\hat{x})\leq D} I(X;\hat{X}).$$

- Recall that $I(X;\hat{X})$ is convex; the problem is a minimization of a convex function over the convex set of all $q(x \mid \hat{x}) \geq 0$ satisfying the constraints
  - $\sum_{\hat{x}} q(\hat{x} \mid x) = 1$ for all $x$,
  - $\sum_{(x,\hat{x})} p(x)q(\hat{x} \mid x)d(x,\hat{x}) \leq D$.
- Reformulate the problem using Lagrange multipliers and we get…

# Characterizing the Rate Distortion Function

Prob Optimize the following functional:

$$J(q) = \sum_{x}\sum_{\hat{x}} p(x)q(\hat{x} \mid x) \log \frac{q(\hat{x} \mid x)}{\sum_x p(x)q(\hat{x} \mid x)}$$

$$+\lambda \sum_{x}\sum_{\hat{x}} p(x)q(\hat{x} \mid x)d(x, \hat{x})$$

$$+\sum_{x} \nu(x) \sum_{\hat{x}} q(\hat{x} \mid x)$$ ........................ [conditional probability]

- We want to know $q(\hat{x}) = \sum_x p(x)q(\hat{x} \mid x)$ values for all $\hat{x} \in \hat{\mathcal{X}}$.

# Characterizing the Rate Distortion Function

**Sol** An optimal solution gives us for all $\hat{x} \in \hat{\mathcal{X}}$,

$$\sum_{x} \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}'} q(\hat{x}')e^{-\lambda d(x,\hat{x}')}} = 1.$$

- From the definition of distortion, we know

$$\sum_{(x,\hat{x})} p(x)q(\hat{x} \mid x)d(x,\hat{x}).$$

- Now we can solve for $q(\hat{x})$ and $\lambda$.
- However, we usually have constraints on $q(\hat{x})$…

# Computing the Rate Distortion Function

**Lem** Let $p(x)p(y \mid x)$ be a given joint distribution. Then,

$$D\big(p(x)p(y \mid x)||p(x)r^*(y)\big) = \min_{r(y)} D\big(p(x)p(y \mid x)||p(x)r(y)\big)$$

where $r^*(y) = \sum_x p(x)p(y \mid x)$.

**Proof** Subtract LHS from $D\big(p(x)p(y \mid x)||p(x)r(y)\big)$ for any $r(y)$ to get $\geq 0$.

# Computing the Rate Distortion Function

Rewrite the rate distortion function (again)

$$R(D) = \min_{q(\hat{x}|x):\sum_{(x,\hat{x})} p(x)q(\hat{x}|x)d(x,\hat{x})\le D} I(X;\hat{X})$$

$$= \min_{q(\hat{x}|x):\sum_{(x,\hat{x})} p(x)q(\hat{x}|x)d(x,\hat{x})\le D} D\big(p(x)q(\hat{x}\mid x)\,||\,p(x)q(\hat{x})\big)$$

Recall that $q(\hat{x}) = \sum_x q(\hat{x},x) = \sum_x p(x)q(\hat{x}\mid x) = r^*(y)$ from the prev. lemma.

$$= \underbrace{\min_{r(\hat{x})}}_{\text{Minimize B}} \underbrace{\min_{q(\hat{x}|x):\sum_{(x,\hat{x})} p(x)q(\hat{x}|x)d(x,\hat{x})\le D}}_{\text{Minimize A}} D\big(p(x)q(\hat{x}\mid x)\,||\,p(x)r(\hat{x})\big)$$

# Computing the Rate Distortion Function

**Alg [Blahut-Arimoto]**

Given: distortion $D$, input distribution $p(x)$ where $X_i$'s are i.i.d. sampled from

Goal: Compute the conditional probability $q(\hat{x} \mid x)$ that minimizes $R(D)$.

- Choose initial $\lambda$ and $|\hat{\mathcal{X}}|$ values $r(\hat{x})$.
- Repeat until convergence:
  - Minimize A by solving for all $(x, \hat{x}) \in \mathcal{X} \times \hat{\mathcal{X}}$

$$q(\hat{x} \mid x) = \frac{r(\hat{x})e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} r(\hat{x})e^{-\lambda d(x,\hat{x})}}$$

Optimization w/ Lagrangian multiplier

  - Minimize B by computing for all $\hat{x} \in \hat{\mathcal{X}}$

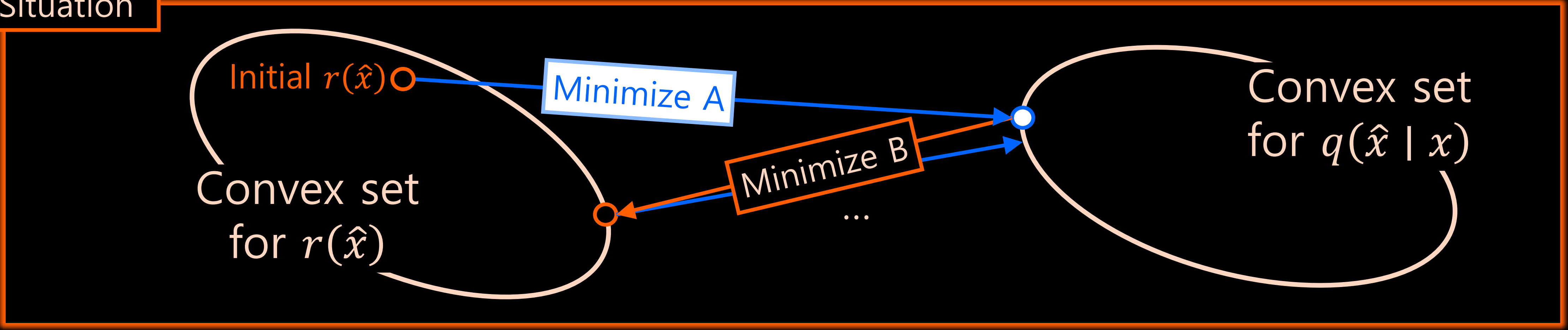
$$r(\hat{x}) = \sum_{x} p(x)q(\hat{x} \mid x)$$

Lemma

# Computing the Rate Distortion Function

Thm [Csiszár] The Blahut-Arimoto algorithm converges to a distribution that gives rate $R(D)$.

Situation




Remark

- Higher $\lambda$ means less compression
- Similar algorithm used for computing channel capacity

# Some Final Remarks

**Thm 10.4.1** For a discrete memoryless channel with capacity $C$, distortion rate $D$ is achievable if and only if $C > R(D)$.

**Thm 10.3.1** The rate distortion function of a Bernoulli$(p)$ source w/ Hamming distortion is given by

$$R(D) = \begin{cases} H(p) - H(D), & 0 \le D \le \min\{p, 1-p\}, \\ 0, & D > \min\{p, 1-p\}. \end{cases}$$

**Thm 10.3.2** The rate distortion function of a $\mathcal{N}(0, \sigma^2)$ source w/ squared-error distortion is given by

$$R(D) = \begin{cases} \frac{1}{2}\log\frac{\sigma^2}{D}, & 0 \le D \le \sigma^2, \\ 0, & D > \sigma^2. \end{cases}$$

# Things to Discuss

- No consensus on a "good" distortion metric for human perception
- What if the input is not i.i.d. sampled from $\mathcal{X}$?

# References


- Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*, 2nd edition. Wiley, 2006.