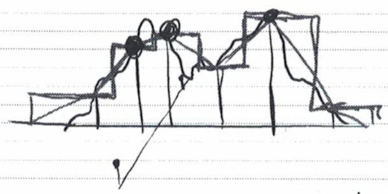
Delaunay Triangulation

2024년 9월 28일 토요일 오후 5:05

Motivation : model a terrain



- a set of sample points P
- we know $f(p)$ $\forall p \in P$.

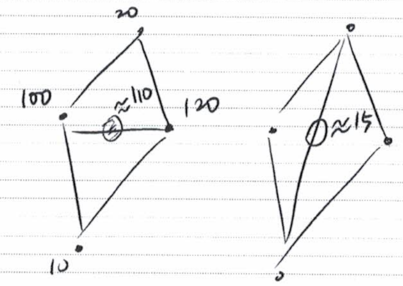
Approach 1.

$f(p') = f(p)$ if $p' \in \underbrace{V(p)}_{\text{Voronoi Cell of } p}$

Approach 2

triangulation! How?

Angle matters.



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a set of n points $P \subseteq \mathbb{R}^2$

(A triangulation of P is a maximal subdivision whose vertex set is P. (Every face is a triangle.) (~~bounded~~))

Let $\mathcal{T}$ be a triangulation of P.

Let m be # $\triangle$ in $\mathcal{T}$.

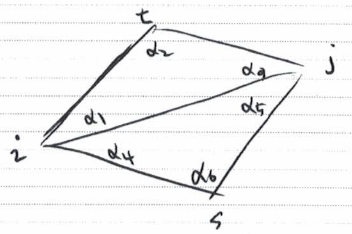
$\text{angle}(\mathcal{T}) = (\alpha_1, \cdots, \alpha_{3m})$

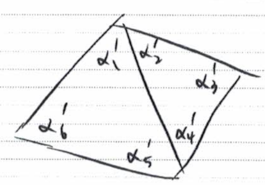
where $\alpha_1 \le \cdots \le \alpha_{3m}$

$\mathcal{T}$ is angle-optimal if $\forall \mathcal{T}'$,

$\text{angle}(\mathcal{T}) \underset{\text{lexicographical}}{\geq} \text{angle}(\mathcal{T}')$

$(3, 3, 4, 5, 5, 6)$
$\qquad\qquad\quad\ \ \geq$
$(3, 3, 4, 4, 9, 10)$





Consider $\triangle ijs$, $\triangle ijt$. (incident)

Sps $\square isjt$ is convex.

- $\mathcal{T}' := \mathcal{T} - \overline{ij} + \overline{st}$ is a triangulation. (edge flip) (abuse of notation)
- We say $\overline{ij}$ is illegal if

$\min_{1 \le l \le 6} \alpha_l < \min_{1 \le l \le 6} \alpha_l'$; o/w legal.

Q. If ($\square ijst$ is convex and) $\overline{ij}$ is illegal, $\text{angle}(\mathcal{T}') > \text{angle}(\mathcal{T})$? Yes

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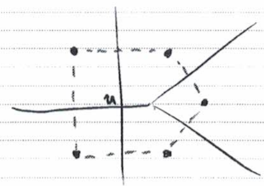
We say $\mathcal{T}$ is legal if $\nexists$ an illegal edge.

Q. If $\mathcal{T}$ is angle-optimal, then is $\mathcal{T}$ legal? Yes.

(Obs. If a legal triangulation is unique, legal $\Leftrightarrow$ angle-optimal.)

DG(P) is a Delaunay Graph of P where P is a node set and for $i,j \in P$, $\overline{ij}$ exists iff $V(i)$ shares an edge w/ $V(j)$ in Vor(P).

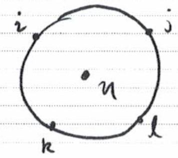


vertex in Vor(P) $\Longleftrightarrow$ face in DG(P)

points on a circle $\Longleftrightarrow$ vertices of the corresponding face of a vertex

[Two properties of Vor(P)]

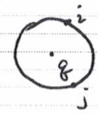
(i) u is a vertex of Vor(P) $\Longleftrightarrow$



largest empty cycle centered at p has at least 3 points on the boundary

(ii) bisec(i, j) contributes to an edge of Vor(P) $\Longleftrightarrow$

$\exists\, q \in$ bisec(i,j) s.t.

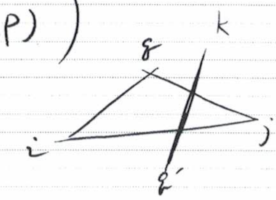


only i, j on the boundary

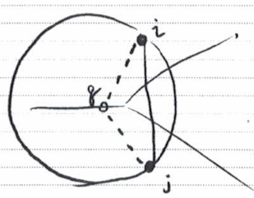
<u>Thm.</u> (If P is a planar point set,) DG(P) is planar

(- trivial if all edges of DG(P) goes through the share edge in Vor(P))

- Sps not. Sps t.c. $\overline{ij}$ $\overline{kl}$ intersects



By (ii),



no point in the circle C

$\Rightarrow k, l \notin C \Rightarrow k, l \notin \Delta qij$

$\Rightarrow \overline{kl}$ intersects $\overline{qi}$ or $\overline{qj}$.

Note $\overline{qi} \subseteq V(i)$ and $\overline{qj} \subseteq V(j)$.

Similarly,

$\Rightarrow \overline{ij}$ intersects $\overline{q'k}$ or $\overline{q'l}$

$\overline{q'k} \subseteq V(k)$ & $\overline{q'l} \subseteq V(l)$

<u>Delaunay triangulation</u> is any triangulation obtained by adding edges to DG(P)

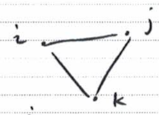
(Obtaining it is easy since all faces of DG(P) are convex)

If no four points on a circle, DG(P) is a Delaunay triangulation.

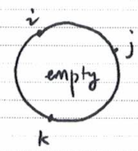
(<u>Obs</u>. DT(P) is unique $\Longleftrightarrow$ P is in "general position")

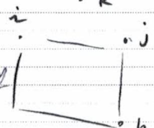
[Two properties] (P: points in the plane)

(i) DG(P) $\bigcirc ijk$ is empty (ii) DG(P)

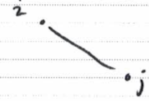


$\Longleftrightarrow$

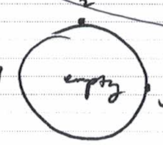




3개의 아무 point를 골라 그것이 face를 구성한다



$\Longleftrightarrow \exists$



$\exists \bigcirc ij$ that is empty

Let $\mathcal{T}$ be a triangulation.

Thm. T is DT(P) $\Longleftrightarrow$ any circumcircle of $\Delta \in \mathcal{T}$ is empty.

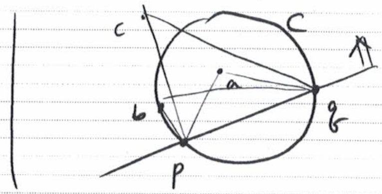
Thm. T is $DT(P) \iff$ any circumcircle of $\Delta \in \mathcal{T}$ is empty.

Q. If $\mathcal{T}$ is $DT(P)$, then is $\mathcal{T}$ legal?

Lemma Consider two incident triangles Δijk and Δijl.

$\overline{ij}$ is illegal $\iff$ $l \in \bigcirc ijk$.

pf) Fact (Thales thm)



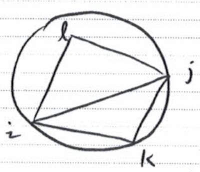
$a \in$ int of C

$b \in$ boundary of C

$d \in$ outside of C

$\measuredangle paq > \measuredangle pbq > \measuredangle pcq$

1) $\Leftarrow$ part.



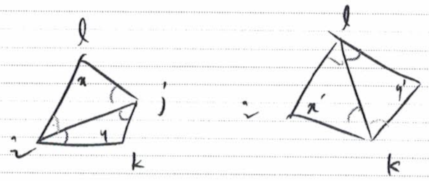
1) Consider $l = \overline{jk}$. $\measuredangle jlk > \measuredangle jik$

$l = \overline{ik}$. $\measuredangle ilk > \measuredangle ijk$

Consider $C' = \bigcirc ijl$. $k \in C'$.

$l = \overline{jl}$. $\measuredangle jkl > \measuredangle jil$

$l = \overline{il}$ $\measuredangle ikl > \measuredangle ijl$



min strictly increases

some angle < min angle

2) Symmetric proof for $\Rightarrow$ ■

Yes. If $\mathcal{T}$ is $DT(P)$, any circumcircle of $\Delta \in \mathcal{T}$ is empty.

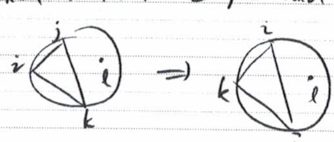
$\Rightarrow$ every common edge is legal.

Thm. Converse is also true.

pf) Sps t.c. that $\mathcal{T}$ is legal but not DT.

$\Rightarrow \exists\, \Delta ijk$ s.t. $l \in \bigcirc ijk$. (not empty)

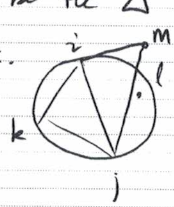
(i, j, k are re-indexed so that Δijk and Δijl do not intersect.)



Among all such tuples $(i\ j\ k\ l)$, pick the one w/ largest $\measuredangle ilj$.

Let Δijm be the Δ incident to Δijk. (It always exists) in order for $\mathcal{T}$ to be a triangulation

Then, $m \notin \bigcirc ijk$.

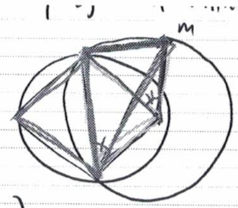


Consider $\bigcirc ijm$. This contains all part of $\bigcirc ijk$ that is separated by $\overline{ij}$ and contains k on the boundary.

$\Rightarrow$ ...

on the boundary.

$\Rightarrow l \in \bigcirc ijm.$



Note, $l \notin \Delta ijm$ by the defn of triangulation.

(again, re-index i, j, m s.t. Δijm and Δjml do not intersects)

By the Fact, $\angle ilm > \angle ijm$ and it is > 0. ↯.

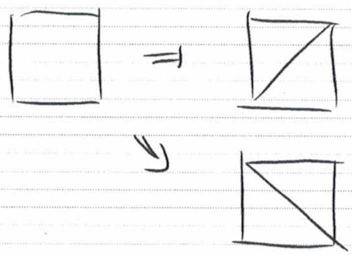
$\Rightarrow$ If $\mathcal{T}$ is angle-optimal, $\mathcal{T}$ is $DT(P)$.

- If P is in general position, $DT(P)$ is unique.

$\Rightarrow$ legal $\mathcal{T}$ is unique $\Rightarrow$ legal $\Leftrightarrow$ angle-optimal.

Q If not in general pos, is $DT(P)$ angle-optimal?

No. But every possible DT has same min angle.



min angle in any triangulation of co-circular points is same.

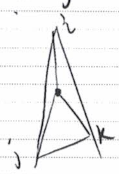
$\Rightarrow$ min angle is independent of the triangulation.

DT(p) maximizes the min angle over all triangulation

- Plane Sweep Voronoi ALG $\rightarrow$ Delaunay Tri ALG
- Plane Sweep for Delauny Tri
- Random Order Insertion (Randomized Incremental)

- $\mathcal{T} \leftarrow \{$ a big triangle $\Delta p_0 p_{-1} p_{-2} \}$
- Sample a permutation π of P u.a.r. $\qquad \pi^{(r)} := \{\pi_1, \ldots, \pi_r\}$
- for $r = 1, \ldots, n$ $\longrightarrow$ here, we have $\mathcal{T} = DT(B \cup \pi^{(r-1)})$

  let $\Delta ijk \in \mathcal{T}$ s.t. $\pi_r \in \Delta ijk$. $\quad$ insert π_x and make $\mathcal{T} = DT(B \cup \pi^{(r)})$



$\longrightarrow \overline{i\pi_x} \in DG(B \cup \pi^{(r)}) \subseteq DT(\text{—})$

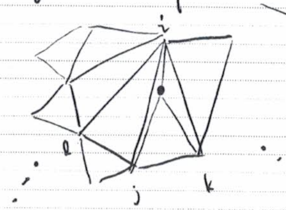
since $\exists \bigcirc ij$ that is empty.

(since $\bigcirc ijk$ was empty before the insertion)

"legalize step" $\longrightarrow$ what could go wrong?

"legalize step" $\longrightarrow$ What could go wrong? (since O_{ijk} was empty before the insertion)

$\overline{ij}, \overline{jk}, \overline{ki}$ can be illegal.



if $\overline{ij}$ is illegal, $\pi_r \in O_{ijl}$.

Since O_{ijl} was empty before the insertion, π_r is the only point inside O_{ijl}.

$\Rightarrow \overline{l\pi_r} \in DG(B \cup \pi^{(r)}) \subseteq DT(\;\cdot\;)$

Q. Will "legalize step" terminate always?

strictly increases the min angle & finite many triangulations.

Assume P is in general position.

Lem. At any iteration r, $E[\#\Delta \text{ created in } r\text{-th iter}] \le O(1)$.

pf) At the end of iter r, let d_r be the degree of π_r.

$\#\Delta$ created $= 2(d_r - 3) + 3 = 2d_r - 3$ (actually at most $2d_r - 3$ when $\triangleleft \pi_r$)

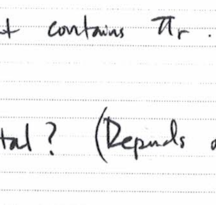
If $E[d_r] \le O(1)$, we are done.

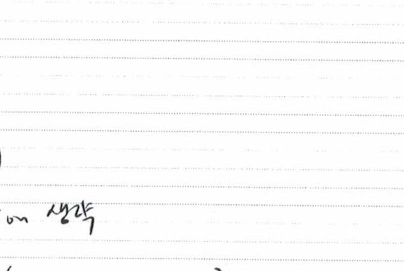
$$\Rightarrow E[d_r \mid \pi^{(r)} = \{p_1, \dots, p_r\}] = \underset{\substack{p \sim u.a.r. \\ p_1, \dots, p_r}}{E}\left[\deg_{DT(p_1,\dots,p_r)}(p)\right] \le O(1)$$

in $Vor(P)$, #edges is $O(n)$
$\Rightarrow$ avg deg is $O(1)$

How to find Δijk that contains π_r?

History (DA) Graph $\mathcal{H}$





$\cup$ "leaf" $= \mathcal{T}$ (current triangulation)

↳ follow the triangle (node) that contains π_r.

How many (expected) #visit's in total? (Depends on π.)

expectation 생각

$$E_\pi\left(\sum_{r=1..n} (\text{"visit" incurred by } \pi_r)\right) = \sum_{\Delta \in \mathcal{H}} (\text{"visit" to } \Delta) \le \sum_{\Delta \in \mathcal{H}} |\textcircled{\Delta} \cap P|$$

Let $\mathcal{T}_r$ be the DT at the end of r-th iter

$$\Delta\in\mathcal{T} \qquad\qquad \Delta\in\mathcal{T} \Big)$$

Let $\mathcal{T}_r$ be the DT at the end of r-th iter.

$$= \sum_{r=1}^{n} \Big(\underbrace{\sum_{\Delta\in\mathcal{T}_r\setminus\mathcal{T}_{r-1}} |\Delta\cap P|} \Big)$$

$$= \sum_{\substack{\Delta\in\mathcal{T}_r : \\ \Delta^{\pi_r}}} |\Delta\cap(P\setminus\pi^{(r)})| = \sum_{q\in P\setminus\pi^{(r)}} \big|\{\Delta\in\mathcal{T}_r : \overset{\pi_r}{\Delta} \text{ and } q\in\Delta\}\big|$$

$$E\Big[\sum_{q\in P\setminus\pi^{(r)}} \big|\{\Delta\in\mathcal{T}_r : \overset{\pi_r}{\Delta} \text{ and } q\in\Delta\}\big| \;\Big|\; \pi^{(r)}=\{p_1,\dots,p_r\}\Big]$$

Δ의 입장에서

$$\underset{(\le)}{=} E\Big[\frac{1}{r}\cdot 3\cdot \underbrace{\sum_{q\in P\setminus\pi^{(r)}} \big|\{\Delta\in\mathcal{T}_r : q\in\Delta\}\big|} \;\Big|\; \text{———}'' \Big]$$

$$(\;)\cdot\frac{1}{n-r} = \underset{\pi_{r+1}}{E}\big[\underbrace{|\{\Delta\in\mathcal{T}_r : \pi_{r+1}\in\Delta\}|}\big]$$

#Δs that will be "destroyed" by the insertion of π_{r+1}.

$$= |\mathcal{T}_r\setminus\mathcal{T}_{r+1}| \overset{\text{obs.}}{=} \underbrace{|\mathcal{T}_{r+1}\setminus\mathcal{T}_r|}_{\text{newly created } \Delta\text{s}} - 2$$

$$\underset{\pi}{E}\Big[\sum_{\Delta\in\mathcal{T}_r\setminus\mathcal{T}_{r-1}} |\Delta\cap P| \;\Big|\; \pi^{(r)} \text{ fixed}\Big] \underset{(\le)}{=} \frac{3}{r}(n-r)\; \underset{\pi}{E}\Big[|\mathcal{T}_{r+1}\setminus\mathcal{T}_r| - 2 \;\Big|\; '' \Big]$$

$$\le O(1)\,\frac{n-r}{r}$$

$$\sum_{r=1}^{n} \Rightarrow O(n\lg n)$$