Quick Recap

Polygon triangulation : $O(n\log n)$ time.

Fortune's alg (Voronoi diagram) : $O(n\log n)$ time.

Incremental Delaunay triangulation : $O(n\log n)$ time.

Range tree w/ fractional cascading : $O(n\log^{d-1} n)$ time construction.

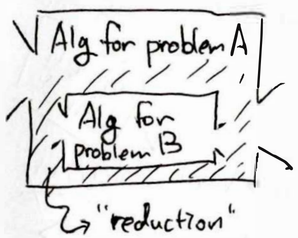
Line segment intersection : $O((n+I)\log n)$ time

>> all of these problems / algs try to improve from $O(n^2)$ sol.

Question What problems cannot be solved in subquadratic time?

Have to show that any alg solving the problem has RT at least $\Omega(n^2)$. ... hard!

Instead, we can show that the problem is at least hard as another problem.



if a $O(n^{2-\varepsilon})$ sol for problem A is unlikely and its "reduction" to problem B is subquadratic, then a $O(n^{2-\varepsilon})$ sol for problem B is unlikely.

Def $A \lll_{f(n)} B$ if problem A can be solved using

- A constant # of instances of problem B of at most linear size
- $O(f(n))$ additional time.

$A ==_{f(n)} B$ if $A \lll_{f(n)} B$ and $B \lll_{f(n)} A$.

Prob 3SUM. ①

Given a set S of n integers, are there $a,b,c \in S$ such that $a+b+c=0$?

Prob 3SUM'

Given three sets A, B, C of integers with $|A|+|B|+|C|=n$, are there $(a,b,c) \in A\times B\times C$ such that $a+b=c$?

Thm 3SUM $==_n$ 3SUM'

i) 3SUM $\lll_n$ 3SUM'

Given $S \rightarrow$ let $A=S$, $B=S$, $C=-S$.

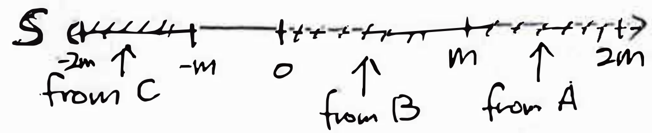
ii) 3SUM' $\lll_n$ 3SUM

Given $A, B, C \rightarrow$ let $k = |\min\{A \cup B \cup \frac{C}{2}\}|$

Add k to A & B, $2k$ to $C \Rightarrow$ w.l.o.g. assume $x \geq 0$ $\forall x \in A\cup B\cup C$.

let $m = \max(A\cup B\cup C)\times 2$.

Construct $S = (A+m) \cup (B) \cup (-C-m)$.



Alg 3SUM'

Sort B & C. ← $O(n\log n)$

for $a \in A$ ← n iters

  $B_a \leftarrow B + a$. ← linear time using two pointers] $O(n^2)$

  if $B_a \cap C \neq \emptyset$, return true.

return false

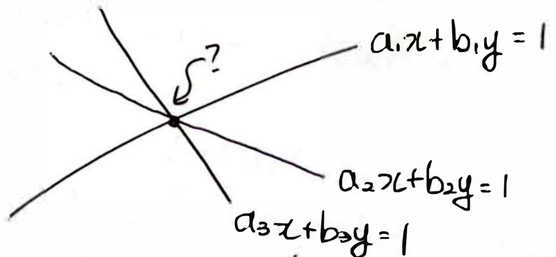
Conjecture Any deterministic alg for 3SUM under the RAM model requires $\Omega(n^2)$ time.

cf) Grønlund and Pettie '14 proved that 3SUM can be solved in $O(n^2/(\log n/\log\log n)^{2/3})$ time under the decision tree model.

Def Problem A is 3SUM-hard iff $\text{3SUM} \lll_{f(n)} A$ and $f(n) \in o(n^2)$.

Prob Point on 3 lines.

Given a set of lines in the plane, is there a point that lies on at least three of them?



Immediate solution: use naive plane sweeping $O(n^2)$

Thm $\text{3SUM} \lll_{n\log n}$ Point on 3 lines

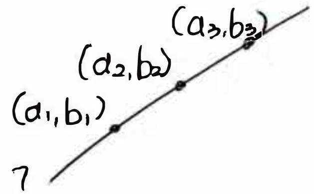
prove by showing

"Dual"

$\boxed{\text{3SUM}} \lll_n \boxed{\text{3 points on line}} ==_{n\log n} \boxed{\text{Point on 3 lines}}$

Prob 3 points on line.

Given a set of points in the plane, is there a line that contains 3+ points?



Observe1 a non-origin point $\langle a,b\rangle$ can be mapped to the line $\langle ax+by=1\rangle$

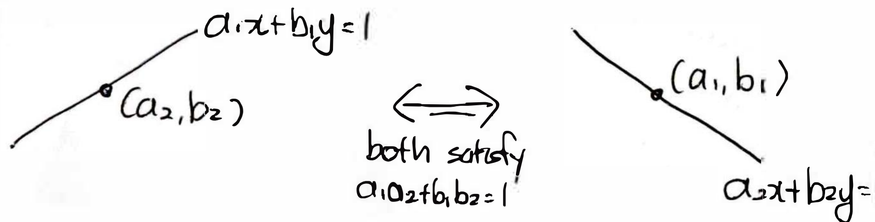
Observe 2 a line $\langle ax+by=1\rangle$ can be mapped to the point $\langle a, b\rangle$

Def A point-line duality D is a mapping that maps points to lines and lines to points, where incidence between points and lines are preserved.

Cornell Demo!

Def polarity. A PL-duality D is called a polarity if $D(\langle ax+by=1\rangle) = \langle a,b\rangle$ and $D(\langle a,b\rangle) = \langle ax+by=1\rangle$

Obs. The polarity function preserves incidence.



Note. The polarity function cannot handle $\langle 0,0\rangle$ nor lines of the form $\langle ax+by=0\rangle$. $\leadsto$ calls for exceptions

Lem 3 points on line $=\!=_{n\log n}$ Point on 3 lines.

i) 3PoL $\lll_{n\log n}$ Po3L.

if $\langle 0,0\rangle \in S$, then identify the slope btwn $\langle 0,0\rangle$ & other points in S. If two pairs share a slope, then return true.

O/w, use duality to map to Po3L.

ii) Po3L $\lll_{n\log n}$ 3PoL.

if $\langle 0,0\rangle$ is on 3+ lines, then return true.

o/w, ≤ 2 lines that cross $\langle 0,0\rangle$. Brute force them.

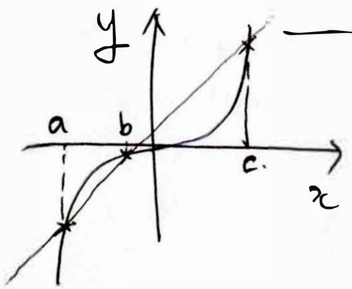
~~Change~~

if no points coincide, remove these lines & use duality.

3SUM $\lll_n$ 3 points on line.

For all $x \in S$, construct 3 points on line:

put $\langle x, x^3\rangle$ in S'



$$y = \frac{a^3-b^3}{a-b}(x-a)+a^3$$

plug in $\langle c, c^3\rangle$:

$$c^3 = \frac{(a-b)(a^2+ab+b^2)}{a-b}(c-a)+a^3$$

$$\Leftrightarrow$$

$$(c-a)(c^2+ac+a^2) = (c-a)(a^2+ab+b^2)$$

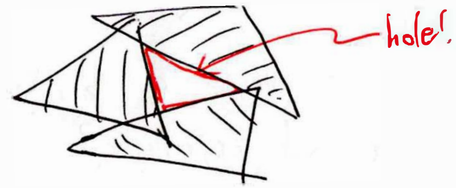
$$c^2-b^2 = -a(c-b)$$

$$a+b+c=0$$

Cor Point on 3 lines is 3SUM-hard.

Prob Hole in union.

Given a set of triangles in the plane, does their union contain a hole?



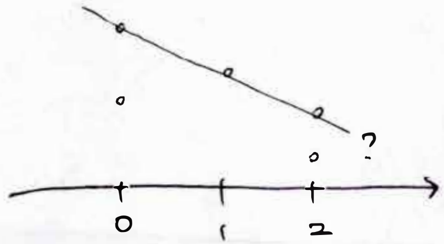
Thm Hole in union is 3SUM-hard.

$\boxed{\text{3SUM}} =\!=_n \boxed{\text{Geom Base}} \lll_{n\log n} \boxed{\text{Strips Cover Box}}$ prove later

$\lll_n \boxed{\text{Triangles cover triangles}} \lll_n \boxed{\text{Hole in union}}$

Prob Geom Base.

Given a set of n points w/ integer coordinates on three vertical lines $x=0$, $x=1$, $x=2$, determine whether there exists a non-vertical line containing three points.



Lem GeomBase $==_n$ 3SUM'

For sets A, B, C, construct the GeomBase set S:

- $(a, a) \in S$ for $a \in A$
- $(1, c/2) \in S$ for $c \in C$
- $(2, b) \in S$ for $b \in B$

$\Rightarrow$ 3 points on a line iff $a+b=c$

Prob Strips cover box.

Given a set of strips in the plane, does their union contain a given axis-parallel rectangle?

Prob Triangles cover triangle.

Given a set of triangles in the plane, does their union contain another given triangle?

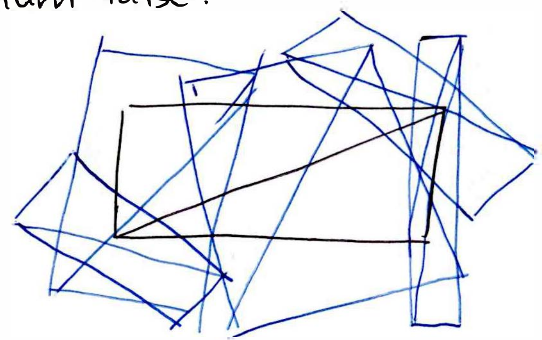
Lem Strips cover box $\lll_n$ Triangles cover triangle

Split the box into two triangles t_1, t_2

Truncate strips into rectangles & split them into triangles T

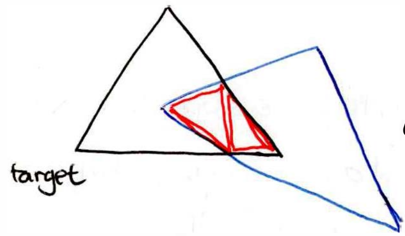
if the triangles in T cover t_1 & t_2, return true

o/w, return false.



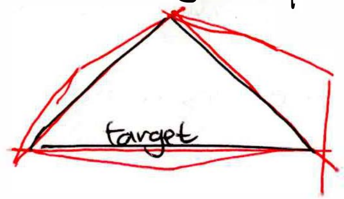
Lem Triangles cover triangle $\lll_n$ Hole in union

For each triangle $t \in T$, intersect with target triangle & split that into triangles, put in S



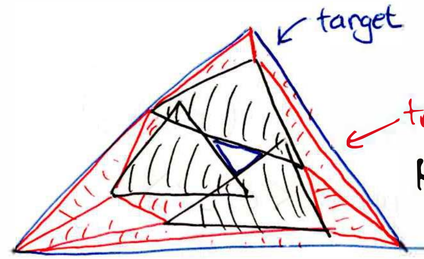
S has triangles strictly inside the target

Make three triangles representing the border of the target, put in S



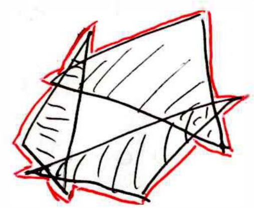
Now S has a hole in its union iff triangles in T do not cover the target.

Remark. Hole in union $\lll_{n \log^2 n}$ Triangles cover triangle.



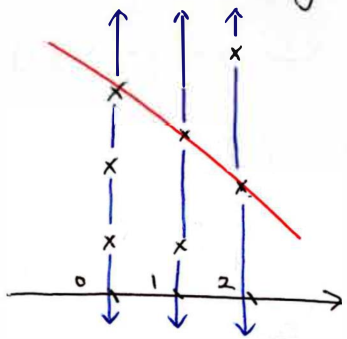
Hole in union iff target not covered

The $n \log^2 n$ bound comes from finding the outer contour of the triangles in T.



Remains to show GeomBase $\ll_{n\log n}$ Strips cover box.

Observe



Let GeomBase consist of $\langle 0, a_i\rangle, \langle 1, b_j\rangle, \langle 2, c_k\rangle$'s where a_i, b_j, c_k are each sorted in incr. order.

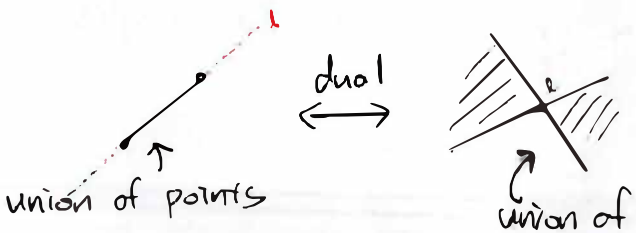
There is a non-vertical line that do not intersect the segments (a_i, a_{i+1}), (b_j, b_{j+1}), (c_k, c_{k+1}) and the half-lines $(-\infty, a_1)$, $(a_{i\max}, \infty)$, $(-\infty, b_1)$, $(b_{j\max}, \infty)$, $(-\infty, c_1)$, $(c_{k\max}, \infty)$

iff GeomBase is a YES instance.

Idea Use duality to map $\begin{cases}\text{lines to points inside box}\\ \text{segments to strips}\end{cases}$

such that GeomBase is YES iff $\exists$ point $\in$ box that is not "incident" to any strip.

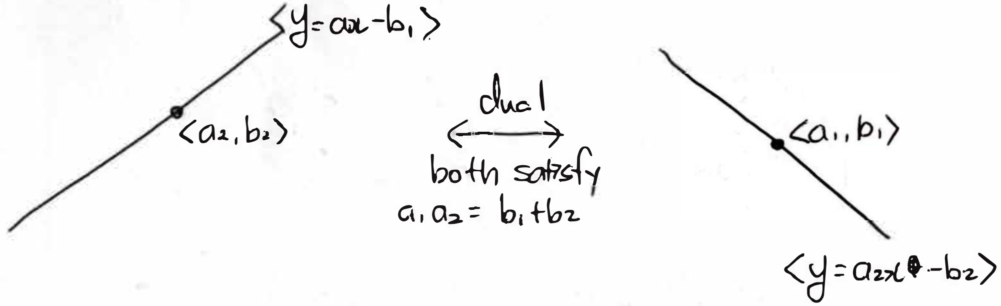
Recall Polarity maps axis-parallel line segments to double wedges



We need a different duality

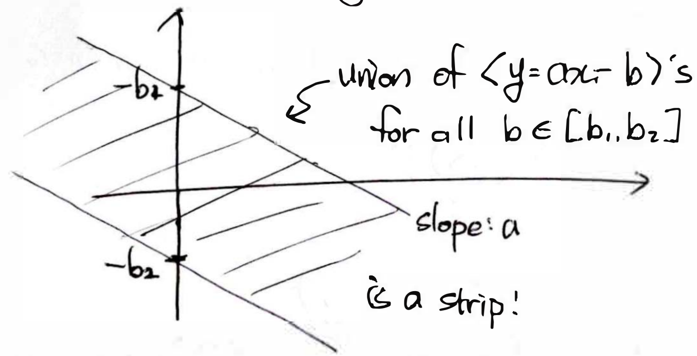
Consider the mapping D:

$$D(\langle a,b\rangle) = \langle y = ax - b\rangle$$
$$D(\langle y = ax - b\rangle) = \langle a, b\rangle$$

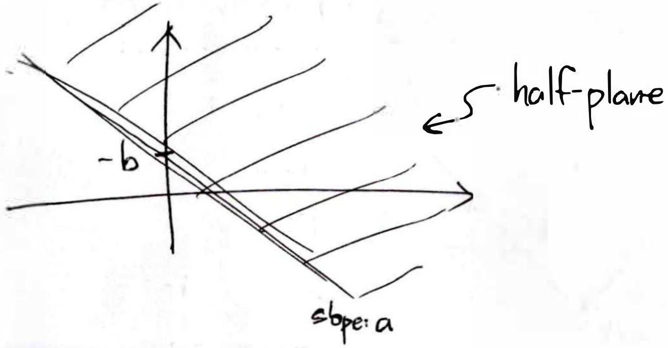


$\rightsquigarrow$ D is a point-line duality.

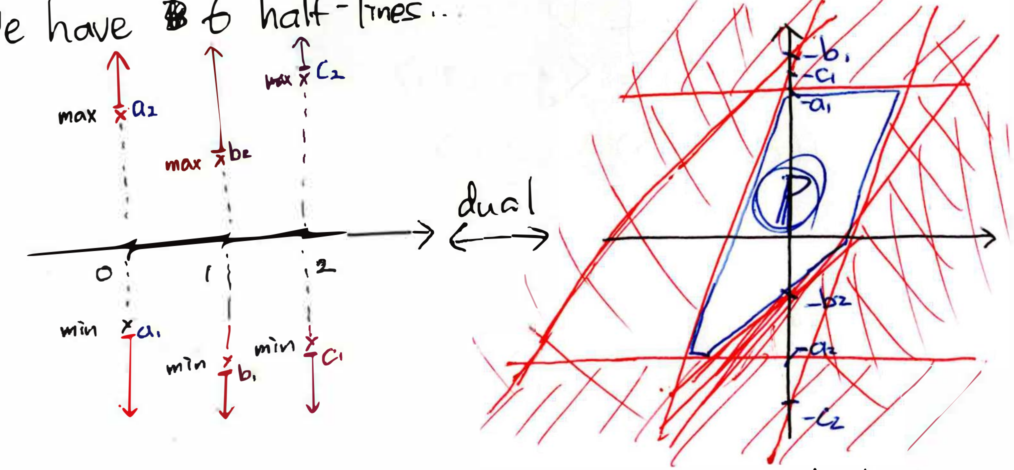
Consider the vertical segment $\langle a, b_1\rangle \sim \langle a, b_2\rangle$.



Or the half-line $\langle a, -\infty\rangle \sim \langle a, b\rangle$



We have 6 half-lines..



GeomBase is YES iff polygon P is covered by dual strips of the line segments.

⇒ <u>Alg</u> GeomBase

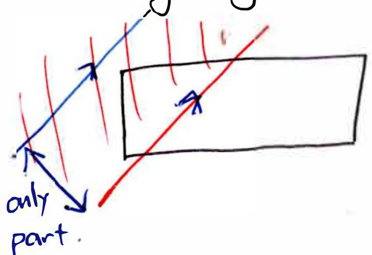
compute the half-lines & segments

$$P \leftarrow \bigcap_{\substack{\text{half line} \\ h}} (D(\langle h \rangle))^c$$

$R \leftarrow$ axis-parallel box containing P

$S \leftarrow$ the set of the duals of all segments

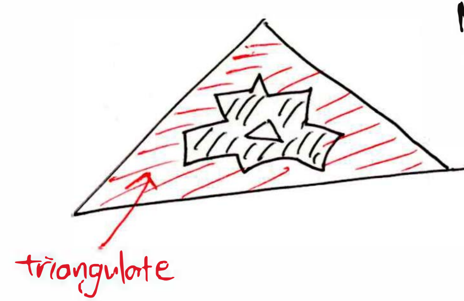
$S \leftarrow S \cup$ (the set of "big enough" strips cut off from half-planes)



Run Strips cover box w/ box R, strip set S & report result.

<u>Thm</u> Hole in union, Triangles cover triangle, Strips cover box, GeomBase are 3SUM-hard.

Remark. We can also consider <u>Triangle measure</u>, which is to compute the measure of the union of triangles.



Measure of large triangle = measure of union iff no hole is in the union.

Remark 2. Recall Klee's measure problem in 2dim.

Bentley's sweeping alg runs in $\Theta(n\log n)$ time

⇒ measuring rectangle union may be easier than measuring triangle union.

On the other hand, 2DKlee $\lll_n$ Triangle measure since we can split n rectangles into $2n$ triangles.