Quantum Max-Cut

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Slides mainly based on Hwang et al. (2023, SODA)
“Unique Games hardness of Quantum Max-Cut, and a conjectured vector-valued Borell’s inequality”

Quantum Basics

Presented by Changyeol Lee

Quantum basics

Qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \binom{\alpha}{\beta}$ where $\alpha, \beta \in \mathbb{C}$ such that $\alpha\alpha^* + \beta\beta^* = 1$.

- $\langle\psi|\psi\rangle = \langle\psi||\psi\rangle = (\alpha^*, \beta^*)\binom{\alpha}{\beta} = \alpha\alpha^* + \beta\beta^* = 1$
- $|\psi\rangle\langle\psi| = \binom{\alpha}{\beta}(\alpha^*, \beta^*) = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix} \quad (tr[|\psi\rangle\langle\psi|] = 1)$

Tensor product

$$|\psi_1\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \binom{\alpha_2}{\beta_2} \\ \beta_1 \binom{\alpha_2}{\beta_2} \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} A_{00}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{01}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \\ A_{01}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} & A_{11}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \end{pmatrix}$$

Hermitian matrix $M = M^*$

- $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$

Quantum basics

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

Fact. For any single qubit state $|\psi\rangle$, the matrix $|\psi\rangle\langle\psi|$ can be uniquely written as $\frac{1}{2}(I + c_X X + c_Y Y + c_Z Z)$ where (c_X, c_Y, c_Z) is on the unit sphere in $\mathbb{R}^3$.

proof)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(I + Z), \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(X + iY), \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}(X - iY), \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(I - Z)$$

$$\frac{1}{2}\big(\alpha\alpha^*(I + Z) + \alpha\beta^*(X + iY) + \beta\alpha^*(X - iY) + \beta\beta^*(I - Z)\big)$$

$$= \frac{1}{2}\big((\alpha\alpha^* + \beta\beta^*)I + (\alpha\beta^* + \beta\alpha^*)X + (i\alpha\beta^* - i\beta\alpha^*)Y + (\alpha\alpha^* - \beta\beta^*)Z\big)$$

$$= \frac{1}{2}(I + c_X X + c_Y Y + c_Z Z)$$

Observe $(c_X, c_Y, c_Z) \in \mathbb{R}^3$ and $\|(c_X, c_Y, c_Z)\| = 1$.

Quantum basics

Fact. For any single qubit state $|\psi\rangle$, the matrix $|\psi\rangle\langle\psi|$ can be uniquely written as $\frac{1}{2}(I + c_X X + c_Y Y + c_Z Z)$ where (c_X, c_Y, c_Z) is on the unit sphere in $\mathbb{R}^3$.

We will refer to this the vector (c_X, c_Y, c_Z) ***Bloch vector*** for $|\psi\rangle$.

Background and Motivation

Some background...

Hamiltonian of a system

$\in \mathbb{R}$ (obtainable from a measurement)

An operator (or Hermitian matrix) s.t. each eigenvalue = one possible value of the system's total energy.

k*-local Hamiltonian *H

A Hermitian matrix acting on *n* qubits which is Σ(Hamiltonian Terms), each acting upon at most *k* qubits.

***k*-local Hamiltonian problem**

Given a *k*-local Hamiltonian *H*, find the smallest eigenvalue λ of *H* (= minimum energy of *H*)

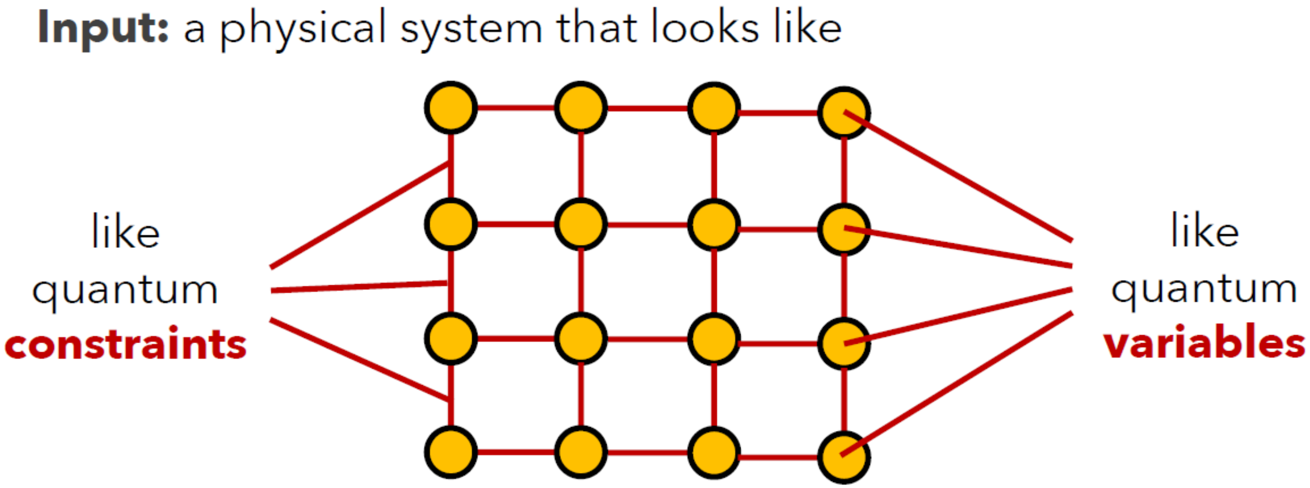
“Quantum analogue of *k*-CSPs (constraint satisfaction problems)”

Some background...

2-local Hamiltonian problem

Given *H*=Σ(Hamiltonian Terms), each acting upon *2* qubits, find λ_{min}.

→ Given a graph with *n* vertices (≈ qubits) and *m* edges (≈ Hamiltonian terms), find λ_{min}.



Slide from John Wright (UC Berkeley)

$$H|\psi\rangle = \lambda|\psi\rangle$$

$$\langle\psi|H|\psi\rangle = \langle\psi|\lambda|\psi\rangle$$

$$\langle\psi|H|\psi\rangle = \lambda\langle\psi|\psi\rangle$$

$$\langle\psi|H|\psi\rangle = \lambda$$

Proposition. $\lambda_{min}(H) = \min_{n \text{ qubits state } |\psi\rangle} \langle\psi|H|\psi\rangle$. (= *expectation value* of H)

Some background...

(Quantum) Heisenberg model

A family of 2-local Hamiltonians first studied by Heisenberg (1928).

The anti-ferromagnetic Heisenberg XYZ model

Given a system G,

Each Hamiltonian Term acts on 2 qubits u and v

There are m number of Hamiltonian Terms

$$H_G^{\text{HEIS}} := \sum_{(u,v)\in E} [X_u \otimes X_v + Y_u \otimes Y_v + Z_u \otimes Z_v] \otimes I_{V\setminus\{u,v\}}$$

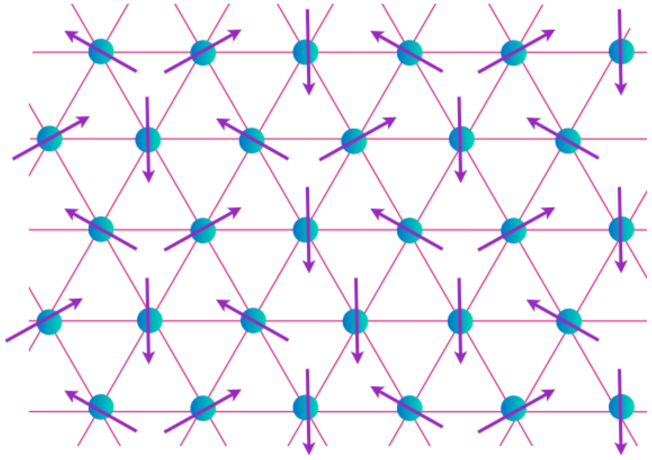
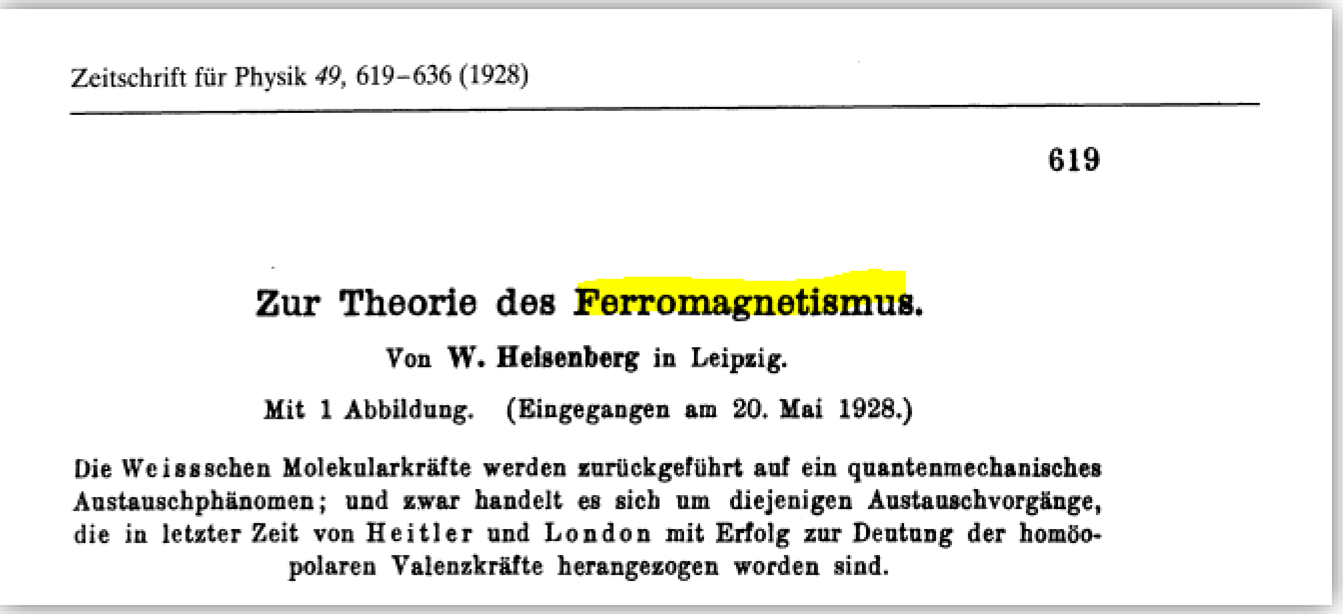


Fig. 2. Ground state of the Heisenberg antiferromagnet on the triangular lattice with long-range antiferromagnetic order. This state is not an example of gapped quantum matter.

Figure from [arXiv:1203.4565v4]



Zeitschrift für Physik 49, 619–636 (1928)

619

Zur Theorie des Ferromagnetismus.

Von **W. Heisenberg** in Leipzig.

Mit 1 Abbildung. (Eingegangen am 20. Mai 1928.)

Die Weissschen Molekularkräfte werden zurückgeführt auf ein quantenmechanisches Austauschphänomen; und zwar handelt es sich um diejenigen Austauschvorgänge, die in letzter Zeit von Heitler und London mit Erfolg zur Deutung der homöopolaren Valenzkräfte herangezogen worden sind.

Quantum Max-Cut

A natural maximization version of the anti-ferromagnetic Heisenberg XYZ model.

"Hamiltonian" for Quantum Max-Cut

$$H_G \coloneqq \sum_{(u,v)\in E} \frac{1}{4}[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v]$$

$$H = \frac{1}{4}(I \otimes I - X \otimes X - Y \otimes Y - Z \otimes Z)$$

The objective is to find

$$\lambda_{max}(H_G) = \max_{n \text{ qubits state } |\psi\rangle} \langle\psi|H_G|\psi\rangle$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Max energy state of H_G ≡ Min energy state of H_G^{HEIS}

  However, two variants differ is in their approximability (more details later)

- H_G is an operator, not a quantum gate (i.e., not unitary).
  - And no quantum circuit today!

Max-Cut

Given a graph $G = (V, E)$, a *cut* is a function $f: V \to \{\pm 1\}$.

We say an edge (u, v) is on the cut f iff $f(u) \neq f(v)$ iff $\frac{1}{2}[1 - f(u)f(v)] = 1$.

The *value* of the cut f is #(edges on f) = $\sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]$.

Find the value of the max cut, i.e., find

$$\max_{f:V\to\{\pm 1\}} \sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]$$

Why Quantum "Max-Cut"?

Consider $H' \coloneqq I \otimes I - Z \otimes Z$

$$I \otimes I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z \otimes Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I \otimes I - Z \otimes Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What is a maximum energy state of H'?

$$\langle 00|H'|00\rangle = 0$$

$$\langle 01|H'|01\rangle = 2$$

$$\langle 10|H'|10\rangle = 2$$

$$\langle 11|H'|11\rangle = 0$$

Why Quantum "Max-Cut"?

Consider $H'_G = \frac{1}{2}\sum_{(u,v)\in E}[I_u \otimes I_v - Z_u \otimes Z_v]$.

A state with max energy of H'_G corresponds to max cut of G.

$$\max_{f:V\to\{\pm 1\}} \sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]$$

$$\lambda_{max}\left(\sum_{(u,v)\in E} \frac{1}{2}[I_u \otimes I_v - Z_u \otimes Z_v]\right)$$

Why Quantum "Max-Cut"?

$$H_G \coloneqq \sum_{(u,v)\in E} \frac{1}{4}[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v]$$

- $-Z_u \otimes Z_v$: measure in Z basis, -1 if same, +1 if different

Similarly,

- $-X_u \otimes X_v$: measure in X basis, -1 if same, +1 if different
- $-Y_u \otimes Y_v$: measure in Y basis, -1 if same, +1 if different

Similar to classical Max-Cut in X, Y and Z bases.

Why Quantum “Max-Cut”?

One more analogy…?

$$\max_{f:V\to\{\pm 1\}} \sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)] = \max_{x\in\{\pm 1\}^{|V|}} x^T L_G x$$

where $L_G = D - A$ is the Laplacian matrix of G.

Recall,

$$\lambda_{max}(H_G) = \max_{n \text{ qubits state } |\psi\rangle} \langle\psi|H_G|\psi\rangle$$

Approximability. Quantum Max-Cut vs Heisenberg model

For the Quantum Max-Cut H_G

- 0.498-approx. (Gharibian and Parekh, 2019)
  - outputs a *product state* using *basic* SDP
- 0.531-approx. (Anshu, Gosset and Morenz, 2020)
- 0.533-approx. (Parekh and Thompson, 2020)

…

- 0.584-approx. (Lee, 2024)
  - outputs products of at most 2-qubit states (using level-2 Quantum Lasserre SDP)

For the anti-ferromagnetic Heisenberg XYZ model H_G^{HEIS}

- $O(\log n)$-approx. (Bravyi et al., 2019)
  - outputs a *product state*

Approximability. Max-Cut vs Ising model

For the Max-Cut $\max_{f:V\to\{\pm 1\}} \sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]$

- 0.878-approx. (Goemans and Williamson, 1995)
  - uses *basic* SDP
  - optimal unless P=NP assuming UGC (Unique Game Conjecture)

For the (anti-ferromagnetic) Ising model $\min_{f:V\to\{\pm 1\}} \sum_{(u,v)\in E} [f(u)f(v)]$

- $O(\log n)$-approx. (Charikar and Wirth, 2004)

Max-Cut Algorithm

Goemans and Williamson (1995)

Briët, Oliveira, and Vallentin (2010)

Presented by Changyeol Lee

Max-Cut and SDP relaxation

Let $S^{d-1} \coloneqq \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ be the d-dimensional unit sphere in $\mathbb{R}^d$.

$$\mathrm{MC}(G) \coloneqq \max_{f:V\to S^0} \sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]$$

SDP relaxation of Max-Cut:

$$\mathrm{SDP}_{\mathrm{MC}}(G) \coloneqq \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{2}[1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

Why is it a relaxation?

- Consider any $f: V \to S^0$. Let $f_{\mathrm{SDP}}(u) = [f(u), 0, \dots, 0] \in S^{n-1}$. Clearly, $f(u)f(v) = \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle$.

Thus, we have $\mathrm{SDP}_{\mathrm{MC}}(G) \geq \mathrm{MC}(G)$.

Note. We can find f_{SDP} of value $\mathrm{SDP}_{\mathrm{MC}}(G) - \epsilon$ in time $\mathrm{poly}(n) \cdot \log 1/\epsilon$.

"Hyperplane" rounding $f_{\mathrm{SDP}}: V \to S^{n-1}$ into $f: V \to S^0$

1. Sample a random $1 \times n$ vector (hyperplane) $\boldsymbol{z} = (z_1, \dots, z_n)$ consisting of n i.i.d. standard Gaussians.

2. For each $u \in V$, set $f(u) = \mathrm{sign}(\boldsymbol{z} f_{\mathrm{SDP}}(u)) = \frac{\boldsymbol{z} f_{\mathrm{SDP}}(u)}{\|\boldsymbol{z} f_{\mathrm{SDP}}(u)\|}$

   i.e., project the vector $f_{\mathrm{SDP}}(u)$ onto the hyperplane $\boldsymbol{z}$ and check its sign.

Goemans and Williamson (1995) showed that for each $(u, v) \in E$,

$$\underbrace{\mathbb{E}_{\boldsymbol{z}}\left[\frac{1}{2}\big(1 - f(u)f(v)\big)\right]}_{\text{prob. } (u,v) \text{ on } f} \geq \underbrace{\frac{2 \arccos \rho_{uv}}{\pi(1 - \rho_{uv})}}_{\text{ratio}} \cdot \underbrace{\frac{1}{2}\left(1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle\right)}_{\mathrm{SDP}_{\mathrm{MC}}((u,v))}$$

where $\rho_{uv} := \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle$.

Let $\alpha_{\mathrm{GW}} := \min_{\rho \in [-1,1]} \frac{2 \arccos \rho}{\pi(1-\rho)} > 0.878.$

By linearity of expectation,

$$\mathbb{E}_{\boldsymbol{z}}\left[\sum_{(u,v)\in E} \frac{1}{2}[1 - f(u)f(v)]\right] \geq \alpha_{\mathrm{GW}} \cdot \sum_{(u,v)\in E} \frac{1}{2}[1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle] \underset{\text{ignoring additive error of } \epsilon}{=} \alpha_{\mathrm{GW}} \cdot \mathrm{SDP}_{\mathrm{MC}}(G) > 0.878 \cdot \mathrm{MC}(G)$$

Rank k Max-Cut and SDP relaxation

$$\mathrm{MC}_k(G) := \max_{f:V\to S^{k-1}} \sum_{(u,v)\in E} \frac{1}{2}[1-\langle f(u), f(v)\rangle]$$

$$\mathrm{SDP}_{\mathrm{MC}}(G) := \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{2}[1-\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

"Projection" rounding ($f_{\mathrm{SDP}}: V \to S^{n-1}$ into $f: V \to S^{k-1}$)

- Sample a random $k \times n$ matirx $\boldsymbol{Z}$ consisting of kn i.i.d. standard Gaussians; and $\forall u \in V,\ f(u) = \frac{\boldsymbol{Z} f_{\mathrm{SDP}}(u)}{\|\boldsymbol{Z} f_{\mathrm{SDP}}(u)\|}$.

Briët, Oliveira, and Vallentin (2010) showed

$$\mathbb{E}_{\boldsymbol{Z}}\left[\frac{1}{2}(1-\langle f(u), f(v)\rangle)\right] \geq \frac{1-F^*(k,\rho_{uv})}{1-\rho_{uv}} \cdot \frac{1}{2}(1-\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle)$$

where $F^*(k,\rho) = \frac{2}{k}\left(\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 \rho \cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{k}{2}+1;\rho^2\right)$ where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gaussian hypergeometric function.

Let $\alpha_{\mathrm{BOV}(k)} := \min_{\rho\in[-1,1]} \frac{1-F^*(k,\rho)}{1-\rho}$. $\alpha_{\mathrm{BOV}(1)} = \alpha_{\mathrm{GW}} > 0.878,\ \alpha_{\mathrm{BOV}(2)} > 0.934,\ \alpha_{\mathrm{BOV}(3)} > 0.956,\ \cdots,\ \alpha_{\mathrm{BOV}(n)} = 1$

Quantum Max-Cut Algorithm

Gharibian and Parekh (2019)

Presented by Changyeol Lee

Quantum Max-Cut algorithm and **ansatz**

Given $H_G \coloneqq \sum_{(u,v)\in E} \frac{1}{4}[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v]$,

find $\mathrm{QMC}(G) \coloneqq \max_{n \text{ qubits state } |\psi\rangle} \langle\psi|H_G|\psi\rangle$ or optimal $|\psi\rangle$.

We consider a classical algorithm.

- *Output* a quantum state $|\psi\rangle$ = *Describe* $|\psi\rangle$ classically
- $|\psi\rangle$ must be efficiently describable.

Q. How to design an **ansatz** to obtain a good approximation ratio?

- 0.498-approximation algorithm of Gharibian and Parekh (2019) uses a *product state*.

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

- Subsequent works (with better ratio) uses a products of at most 2-qubit states. E.g.,

$$|\psi\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle \otimes \cdots \otimes |\psi_n\rangle$$ (also efficiently describable)

(entangled)

Quantum Max-Cut algorithm and **product state ansatz**

Given $H_G := \sum_{(u,v)\in E} \frac{1}{4}[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v]$,

find $\mathrm{QMC}(G) := \max_{n \text{ qubits state } |\psi\rangle} \langle \psi | H_G | \psi \rangle$ or optimal $|\psi\rangle$.

We focus on the *product state ansatz*. $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$

The *product state value* of H_G is

$$\mathrm{QMC}_{\mathrm{PROD}}(G) := \max_{\substack{|\psi_G\rangle = \otimes_{v\in V} |\psi_v\rangle : \\ 1 \text{ qubit state } |\psi_v\rangle}} \langle \psi_G | H_G | \psi_G \rangle$$

Somehow, we want to use the projection rounding.

Can we rewrite $\mathrm{QMC}_{\mathrm{PROD}}(G)$ like $\mathrm{MC}_k(G) = \max_{f:V\to S^{k-1}} \sum_{(u,v)\in E} \frac{1}{2}[1 - \langle f(u), f(v) \rangle]$?

Rewriting the product state value

Proposition. $\mathrm{QMC}_{\mathrm{PROD}}(G) = \max_{\substack{|\psi_G\rangle = \otimes_{v\in V}|\psi_v\rangle : \\ 1 \text{ qubit state } |\psi_v\rangle}} \langle \psi_G | H_G | \psi_G \rangle = \max_{\boldsymbol{f}: \boldsymbol{V} \to \boldsymbol{S}^2} \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \boldsymbol{E}} \frac{\mathbf{1}}{\mathbf{4}} [\mathbf{1} - \langle \boldsymbol{f}(\boldsymbol{u}), \boldsymbol{f}(\boldsymbol{v}) \rangle]$.

proof) First observe $\langle \psi_G | H_G | \psi_G \rangle = tr[H_G |\psi_G\rangle\langle\psi_G|] = tr[H_G \otimes_{v\in V} |\psi_v\rangle\langle\psi_v|]$.

For each $v \in V$, let $f(v) = (v_X, v_Y, v_Z) \in S^2$ be the Bloch vector for $|\psi_v\rangle$.

Fix any $(u, v) \in E$.

$$tr\left[\frac{1}{4}[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v] \cdot |\psi_u\rangle\langle\psi_u| \otimes |\psi_v\rangle\langle\psi_v|\right]$$

$$= tr\left[\frac{1}{4}[I \otimes I - X \otimes X - Y \otimes Y - Z \otimes Z] \cdot \frac{1}{2}[I + u_X X + u_Y Y + u_Z Z] \otimes \frac{1}{2}[I + v_X X + v_Y Y + v_Z Z]\right]$$

$$= \frac{1}{4}[1 - u_X v_X - u_Y v_Y - u_Z v_Z]$$

$$= \frac{1}{4}[1 - \langle f(u), f(v) \rangle]$$

- $X^2 = Y^2 = Z^2 = I$
- $XY = iZ, YZ = iX,$ and $ZX = iY$
- $XY = -YX, YZ = -ZY,$ and $ZX = -XZ$
- $tr[X] = tr[Y] = tr[Z] = 0$

Quantum Max-Cut algorithm and product state ansatz

$$\mathrm{QMC}_{\mathrm{PROD}}(G) \coloneqq \max_{f: V \to S^2} \sum_{(u,v) \in E} \frac{1}{4}[1 - \langle f(u), f(v) \rangle]$$

Note. $\mathrm{QMC}_{\mathrm{PROD}}(G) \leq \frac{1}{2} \cdot |E|$ always (even when $\mathrm{QMC}(G)$ can be as large as $|E|$).

Observation. $\mathrm{QMC}_{\mathrm{PROD}}(G) = \frac{1}{2} \cdot \mathrm{MC}_3(G)$.

Can we apply the algorithm of Briët, Oliveira, and Vallentin (2010) for Rank 3 Max-Cut?

Quantum Max-Cut algorithm and product state ansatz

$$\mathrm{QMC}_{\mathrm{PROD}}(G) \coloneqq \max_{f:V\to S^2} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]$$

Claim. Similar to Rank 3 Max-Cut, we obtain an algorithm with approximation ratio $\alpha_{\mathrm{BOV}(3)}$.

1. SDP Relaxation of $\mathrm{QMC}(G)$

$$\mathrm{SDP}(G) \coloneqq \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

2. Projection rounding

$$\sum_{(u,v)\in E} \mathbb{E}_{\mathbf{Z}}\left[\frac{1}{4}(1 - \langle f(u), f(v)\rangle)\right] \geq \alpha_{\mathrm{BOV}(3)} \cdot \sum_{(u,v)\in E} \frac{1}{4}(1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle) = \alpha_{\mathrm{BOV}(3)} \cdot \mathrm{SDP}(G)$$

Quantum Max-Cut algorithm and product state ansatz

$$\mathrm{QMC}_{\mathrm{PROD}}(G) \coloneqq \max_{f:V\to S^2} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]$$

Wrong Claim. Similar to Rank 3 Max-Cut, we obtain an algorithm with approximation ratio $\alpha_{\mathrm{BOV}(3)} > 0.956$.

1. SDP Relaxation of $\mathrm{QMC}(G)$

$$\mathrm{SDP}(G) \coloneqq \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

2. Projection rounding

$$\sum_{(u,v)\in E} \mathbb{E}_{\boldsymbol{z}}\left[\frac{1}{4}(1 - \langle f(u), f(v)\rangle)\right] \geq \alpha_{\mathrm{BOV}(3)} \cdot \sum_{(u,v)\in E} \frac{1}{4}(1 - \langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle) = \alpha_{\mathrm{BOV}(3)} \cdot \mathrm{SDP}(G)$$

Quantum Max-Cut algorithm and product state ansatz

$$\mathrm{QMC}_{\mathrm{PROD}}(G) \coloneqq \max_{f:V\to S^2} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]$$

Correct Claim. There is an algorithm that outputs a value $\geq \alpha_{\mathrm{BOV}(3)} \cdot [(\text{best})\ \text{product state value}]$.

To say "there is an algorithm that outputs a value $\geq \alpha \cdot \mathrm{QMC}(G)$" using a similar arguments,

we need <u>a valid relaxation</u>.

SDP relaxation for Quantum Max-Cut

Proposition.

$$\mathrm{SDP_{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}\left[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle\right].$$

Let $|\psi\rangle$ be a n-qubit quantum state. The energy of $|\psi\rangle$ is as follows:

$$\langle\psi|H_G|\psi\rangle = \sum_{(u,v)\in E} \frac{1}{4}\langle\psi|[I_u \otimes I_v - X_u \otimes X_v - Y_u \otimes Y_v - Z_u \otimes Z_v]|\psi\rangle$$

Consider $3n$ number of vectors $\sigma_u|\psi\rangle$ for all $\sigma \in \{X, Y, Z\}$ and for all $u \in V$.

Let M be a $3n \times 3n$ (Gram) matrix whose rows and columns are indexed by σ_u such that

$$M(\sigma_u, \sigma'_v) = \left\langle \sigma_u|\psi\rangle, \sigma'_v|\psi\rangle \right\rangle = \langle\psi|\sigma_u\sigma'_v|\psi\rangle.$$

Then $\langle\psi|H_G|\psi\rangle = \sum_{(u,v)\in E} \frac{1}{4}[1 - M(X_u, X_v) - M(X_u, X_v) - M(X_u, X_v)]$.

→ We write a program that maximizes $\langle\psi|H_G|\psi\rangle$ over all "valid" matrix M.
"Valid relaxation"

SDP relaxation for Quantum Max-Cut

Proposition.

$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}\left[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle\right].$$

Let us derive some constraint that M satisfies.

(1) M is Hermitian and PSD.

(2) $M(\sigma_u, \sigma_u) = 1$ for each σ_u.

(3) $M(\sigma_u, \sigma'_v) = M(\sigma'_v, \sigma_u)$ for each σ_u, σ'_v s.t. $u \neq v$. (Only real part exists.)

(4) $M(\sigma_u, \sigma'_u) = -M(\sigma'_u, \sigma_u)$ for each σ_u, σ'_u s.t. $\sigma \neq \sigma'$. (No real part exists.)

We solve the following optimization problem using SDP:

$$\begin{aligned} &\text{maximize} \quad \mathrm{SDP}_{\mathrm{QMC}}(G) = \sum_{(u,v)\in E} \frac{1}{4}\left[1 - M(X_u, X_v) - M(X_u, X_v) - M(X_u, X_v)\right], \\ &\text{subject to} \qquad\qquad (1) - (4). \end{aligned}$$

SDP relaxation for Quantum Max-Cut

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$$\begin{aligned} &\text{maximize} \quad \mathrm{SDP}_{\mathrm{QMC}}(G) = \sum_{(u,v)\in E} \frac{1}{4}[1 - M(X_u, X_v) - M(X_u, X_v) - M(X_u, X_v)], \\ &\text{subject to} \quad (1) - (4). \end{aligned}$$

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Let us derive some constraint that M satisfies.

(1) M is symmetric and PSD.

(2) $M(\sigma_u, \sigma_u) = 1$ for each σ_u.

~~(3) $M(\sigma_u, \sigma'_v) = M(\sigma'_v, \sigma_u)$ for each σ_u, σ'_v s.t. $u \neq v$. (Only real part exists.)~~

(4) $M(\sigma_u, \sigma'_u) = 0$ for each σ_u, σ'_u s.t. $\sigma \neq \sigma'$. (Trivially, only real part exists.)

We solve the following optimization problem using SDP:

$$\text{maximize} \quad \mathrm{SDP}_{\mathrm{QMC}}(G) = \sum_{(u,v)\in E} \frac{1}{4}[1 - M(X_u, X_v) - M(X_u, X_v) - M(X_u, X_v)],$$

$$\text{subject to} \quad (1), (2) \text{ and } (4).$$

Can we solve this now?

SDP relaxation for Quantum Max-Cut

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$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}: V \to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle].$$

Since M is real, symmetric $3n \times 3n$ PSD matrix, there exists a function $g: V \times \{X, Y, Z\} \to \mathbb{R}^{3n}$ such that

$$M(\sigma_u, \sigma'_v) = \langle g(u,\sigma), g(v,\sigma')\rangle.$$

$$\begin{aligned}
&\text{maximize} \quad \mathrm{SDP}_{\mathrm{QMC}}(G) = \frac{1}{4}\sum_{(u,v)\in E} [1 - \langle g(u,X), g(v,X)\rangle - \langle g(u,Y), g(v,Y)\rangle - \langle g(u,Z), g(v,Z)\rangle],\\
&\text{subject to} \quad \begin{array}{ll} \langle g(u,\sigma), g(u,\sigma')\rangle = 0, & \forall u \in V, \sigma \neq \sigma' \in \{X,Y,Z\},\\ g(\cdot,\sigma): V \to S^{3n-1}, & \forall \sigma \in \{X,Y,Z\}. \end{array}
\end{aligned}$$

SDP relaxation for Quantum Max-Cut

Proposition.

$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}: V \to \boldsymbol{S}^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle].$$

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$$\begin{aligned} \text{maximize} \quad & \mathrm{SDP}_{\mathrm{QMC}}(G) = \frac{1}{4}\sum_{(u,v)\in E} [1 - \langle g(u,X), g(v,X)\rangle - \langle g(u,Y), g(v,Y)\rangle - \langle g(u,Z), g(v,Z)\rangle], \\ \text{subject to} \quad & \langle g(u,\sigma), g(u,\sigma')\rangle = 0, \quad \forall u \in V, \sigma \neq \sigma' \in \{X,Y,Z\}, \\ & g(\cdot,\sigma): V \to \boldsymbol{S}^{3n-1}, \qquad \forall \sigma \in \{X,Y,Z\}. \end{aligned}$$

SDP relaxation for Quantum Max-Cut

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$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}\left[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle\right].$$

LHS $\leq$ RHS

SDP relaxation for Quantum Max-Cut

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LHS ≤ RHS

$$\mathrm{SDP}_{\mathrm{QMC}}(G)$$

$$= \frac{1}{4} \sum_{(u,v)\in E} [1 - \langle g^*(u,X), g^*(v,X)\rangle - \langle g^*(u,Y), g^*(v,Y)\rangle - \langle g^*(u,Z), g^*(v,Z)\rangle]$$

$$= \sum_{(u,v)\in E} \frac{1}{3} \cdot \frac{1}{4}[1 - 3\langle g^*(u,X), g^*(v,X)\rangle] + \frac{1}{3} \cdot \frac{1}{4}[1 - 3\langle g^*(u,Y), g^*(v,Y)\rangle] + \frac{1}{3} \cdot \frac{1}{4}[1 - 3\langle g^*(u,Z), g^*(v,Z)\rangle]$$

since $g^*(\cdot,\sigma): V \to S^{n-1}$

$$\leq \max_{\sigma\in\{X,Y,Z\}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle g^*(u,\sigma), g^*(v,\sigma)\rangle] \leq \max_{f_{\mathrm{SDP}}:V \to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

SDP relaxation for Quantum Max-Cut

Proposition.

$$\mathrm{SDP_{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}\left[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle\right].$$

LHS $\leq$ RHS

RHS $\leq$ LHS

Let $g(u,\sigma_i) \coloneqq e_i \otimes f^*_{\mathrm{SDP}}(v)$ where $\sigma_1 = X, \sigma_2 = Y$ and $\sigma_3 = Z$ and e_1, e_2, e_3 are standard basis of $\mathbb{R}^3$.

g is feasible solution to the program, i.e.,

$$\begin{aligned} &\langle g(u,\sigma), g(u,\sigma')\rangle = 0, && \forall u \in V, \sigma \neq \sigma' \in \{X,Y,Z\}, \\ &g(\cdot,\sigma): V \to S^{3n-1}, && \forall \sigma \in \{X,Y,Z\}. \end{aligned}$$

and its objective value is equal to $\sum_{(u,v)\in E} \frac{1}{4}\left[1 - 3\langle f^*_{\mathrm{SDP}}(u), f^*_{\mathrm{SDP}}(v)\rangle\right]$.

Quantum Max-Cut algorithm

$$\mathrm{QMC}_{\mathrm{PROD}}(G) = \max_{f:V\to S^2} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]$$

$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

(1) Compute an optimal f_{SDP} and

(2) Apply the projection rounding (sample a matrix $\boldsymbol{Z} \sim N(0,1)^{3\times n}$ and $\forall u \in V,\ f(u) = \frac{\boldsymbol{Z} f_{\mathrm{SDP}}(u)}{\|\boldsymbol{Z} f_{\mathrm{SDP}}(u)\|}$)

Gharibian and Parekh (2019) showed, for any edge $(u, v) \in E$,

$$\mathbb{E}_{\boldsymbol{Z}}\left[\frac{1}{4}(1 - \langle f(u), f(v)\rangle)\right] \geq \frac{1 - F^*(3, \rho_{uv})}{1 - 3\rho_{uv}} \cdot \frac{1}{4}(1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle)$$

negative value for $\rho \in [1/3,1]$

Let $\alpha_{\mathrm{GP}} \coloneqq \min_{\rho\in[-1,1/3]} \frac{1-F^*(3,\rho)}{1-3\rho} > 0.498$. By linearity of expectation,

$$\mathbb{E}_{\boldsymbol{Z}}\left[\sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]\right] \geq \alpha_{\mathrm{GP}} \cdot \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle] = \alpha_{\mathrm{GP}} \cdot \mathrm{SDP}_{\mathrm{QMC}}(G) > 0.498 \cdot \mathrm{QMC}(G)$$

Quantum Max-Cut algorithm

$$\mathrm{QMC}_{\mathrm{PROD}}(G) = \max_{f:V\to S^2} \sum_{(u,v)\in E} \frac{1}{4}[1 - \langle f(u), f(v)\rangle]$$

$$\mathrm{SDP}_{\mathrm{QMC}}(G) = \max_{f_{\mathrm{SDP}}:V\to S^{n-1}} \sum_{(u,v)\in E} \frac{1}{4}[1 - 3\langle f_{\mathrm{SDP}}(u), f_{\mathrm{SDP}}(v)\rangle]$$

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Gharibian and Parekh (2019) showed

"the above alg. outputs a product state whose value is at least $\alpha_{\mathrm{GP}} \cdot \mathrm{SDP}_{\mathrm{QMC}}(G)$."

Parekh and Thompson (2022) gives a 0.5-approx. alg. that outputs a product state

that uses 2nd level of the quantum Lasserre hierarchy for H_G.

Hardness of Quantum Max-Cut

Hwang et al. (2023)

Hardness related to Max-Cut

The α_{GW}-approx. alg. of Goemans and Williamson (1995) for Max-Cut

- The *basic* SDP rounding alg.
- Feige and Schechtman (2002) showed the integrality gap of this SDP is α_{GW}
  - It is an optimal basic SDP rounding alg.!
- Khot et al. (2007) showed it is optimal unless P=NP assuming UGC
  - It is an optimal alg.!
  - In particular, *strengthening* SDP does not improve the approx. ratio.

Raghavendra (2008) showed

"Assuming UGC, for each CSP, the "canonical alg." based on the "basic" SDP is optimal unless P=NP."

Hardness related to Quantum Max-Cut

★ vector-valued Borell's inequality

The α_{GP}-approx. alg. of Gharibian and Parekh (2019) for Quantum Max-Cut

- The (*basic* SDP rounding) + (product ansatz) alg.
- Hwang et al. (2023) showed the integrality gap of this SDP is α_{GP} assuming ★
  - It is an optimal among all (basic SDP rounding) + (any ansatz) alg.!
- Strengthening SDP strictly improves the approx. ratio. (Anshu, Gosset and Morenz, 2020)
  - Even when restricted to using the product state (Parekh and Thompson (2022)) assuming ★
    - Opposite to CSP where basic SDP is always optimal under UGC
- Hwang et al. (2023) showed NP-hard to do better than $\alpha_{\mathrm{BOV}(3)}$ assuming ★ and UGC
  - Current best known is 0.584-approx. alg. by Lee (2024)

Borell's inequality

Showing the integrality gap of $\mathrm{SDP}_{\mathrm{MC}}(\cdot)$ is α_{GW}

- Construct a graph G^* such that

1. The hyperplane rounding outputs exactly $\alpha_{\mathrm{GW}} \cdot \mathrm{SDP}_{\mathrm{MC}}(G^*)$.

2. The hyperplane rounding is optimal, i.e., it outputs $\mathrm{MC}(G^*)$.

$$\inf_{\forall G}\left\{\frac{\mathrm{MC}(G)}{\mathrm{SDP}_{\mathrm{MC}}(G)}\right\} \leq \frac{\mathrm{MC}(G^*)}{\mathrm{SDP}_{\mathrm{MC}}(G^*)} = \frac{\alpha_{\mathrm{GW}} \cdot \mathrm{SDP}_{\mathrm{MC}}(G^*)}{\mathrm{SDP}_{\mathrm{MC}}(G^*)} = \alpha_{\mathrm{GW}}$$

proof of step 2)

Recall the hyperplane rounding returns $\frac{1}{2}\left(1 - \mathbb{E}\left[\boldsymbol{r}^*\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}^*\left(f_{\mathrm{SDP}}(v)\right)\right]\right)$ where $\boldsymbol{r}^*\left(f_{\mathrm{SDP}}(\cdot)\right) = \frac{\boldsymbol{z} f_{\mathrm{SDP}}(\cdot)}{\|\boldsymbol{z} f_{\mathrm{SDP}}(\cdot)\|}$.

Given an optimal SDP soln f_{SDP} for $\mathrm{SDP}_{\mathrm{QM}C}(G^*)$, *Borell's (isoperimetric) inequality* gives

$$\mathbb{E}\left[\boldsymbol{r}^*\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}^*\left(f_{\mathrm{SDP}}(v)\right)\right] \leq \mathbb{E}\left[\boldsymbol{r}\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}\left(f_{\mathrm{SDP}}(v)\right)\right]$$

for any rounding $\boldsymbol{r}: \mathbb{R}^n \to \{\pm 1\}$.

Borell’s inequality

Showing the integrality gap of $\mathrm{SDP}_{\mathrm{QMC}}(\cdot)$ is α_{GP}

- Construct a graph G^* such that

1. The (projection rounding) + (product ansatz) outputs exactly $\alpha_{\mathrm{GP}} \cdot \mathrm{SDP}_{\mathrm{QMC}}(G^*)$.

2. The (projection rounding) + (product ansatz) is optimal, i.e., it outputs $\mathrm{QMC}(G^*)$.

proof of step 2)

Recall the (projection rounding) + (product ansatz) $\boldsymbol{r}^*: \mathbb{R}^n \to S^2$ returns $\frac{1}{4}\left(1 - \mathbb{E}\left[\left\langle \boldsymbol{r}^*\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}^*\left(f_{\mathrm{SDP}}(v)\right)\right\rangle\right]\right)$.

Given an optimal SDP soln f_{SDP} for $\mathrm{SDP}_{\mathrm{QMC}}(G^*)$, *vector-valued Borell’s inequality* gives

$$\mathbb{E}\left[\boldsymbol{r}^*\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}^*\left(f_{\mathrm{SDP}}(v)\right)\right] \leq \mathbb{E}\left[\boldsymbol{r}\left(f_{\mathrm{SDP}}(u)\right) \cdot \boldsymbol{r}\left(f_{\mathrm{SDP}}(v)\right)\right]$$

for (any rounding) + (product ansatz) $\boldsymbol{r}: \mathbb{R}^n \to S^2$.

Brandão and Harrow (2016) showed that the product state is (roughly) identical to an optimal state of high-degree graph and G^* is a high-degree graph.

→ For G^*, (projection rounding) + (product ansatz) is optimal among (any rounding) + (any ansatz).

Takeaway

Takeaway

Basics

- $\langle\psi|\psi\rangle = tr[|\psi\rangle\langle\psi|] = 1$
  - $|\psi\rangle\langle\psi|$ is called *density matrix* of $|\psi\rangle$
- 1-qubit $|\psi\rangle$ → *Bloch vector* $(c_X, c_Y, c_Z) \in S^2$

Hamiltonian problem

- the term itself; quantum analogue of CSP
- eigenvalue of $|\psi\rangle = \langle\psi|H|\psi\rangle = tr[H|\psi\rangle\langle\psi|]$
- **Ansatz**. For quantum problem, we can design a classical algorithm that outputs a description of the state.

Quantum Max-Cut

- can design *basic* SDP and apply *standard* rounding technique as in classical Max-Cut
- strengthening SDP does help compared to classical world where basic SDP is optimal for all CSP
  - even when using product state ansatz (assuming ★)

And there are many open questions! (e.g., second level SDP is optimal?)

Thank you.